Approximation algorithms for minimum-cost k-(S, T) connected digraphs

J. Cheriyan * B. Laekhanukit †

January 28, 2012

Abstract

In the *minimum-cost* k-(S, T) *connected digraph* (abbreviated, k-(S, T) connectivity) problem we are given a positive integer k, a directed graph $G = (V, E)$ with non-negative costs on the edges, and two subsets S, T of V; the goal is to find a subset of edges $\widehat{E}$ of minimum cost such that the subgraph $(V, \widehat{E})$ has k edge-disjoint directed paths from each vertex in S to each vertex in T.

Most of our results focus on a specialized version of the problem that we call the *standard* version, where every edge of positive cost has its tail in S and its head in T. This version of the problem captures NP-hard problems such as the minimum-cost k-vertex connected spanning subgraph problem. We give an approximation algorithm with a guarantee of $O((\log k)(\log n))$ for the standard version of the k-(S, T) connectivity problem, where n denotes the number of vertices. For $k = 1$, we give a simple 2-approximation algorithm that generalizes a well-known 2-approximation algorithm for the minimum-cost strongly connected spanning subgraph problem. For $k = 2$, we give a 3-approximation algorithm.

Besides the standard version, we study another version that is intermediate between the standard version and the problem in its full generality. In the relaxed version of the (S, T) connectivity problem, each edge of positive cost has its head in T but there is no restriction on the tail. We study the relaxed version with the connectivity parameter k fixed at one, and observe that this version is at least as hard to approximate as the directed Steiner tree problem. We match this by giving an algorithm that achieves an approximation guarantee of $\alpha(n) + 1$ for the relaxed (S, T) connectivity problem, where $\alpha(n)$ denotes the best approximation guarantee available for the directed Steiner tree problem. The key to the analysis is a structural result that decomposes any feasible solution into a set of so-called junction trees that are disjoint on the vertices of T. Our algorithm and analysis specialize to the case when the digraph is acyclic on T, meaning that there exists no dicycle that contains two distinct vertices of T. In this setting, we show that the relaxed (S, T) connectivity problem is at least as hard to approximate as the set covering problem, and we prove that our algorithm achieves a matching approximation guarantee of $O(\log |S|)$.

Keywords: approximation algorithms, graph connectivity, network design, k-vertex connected spanning subgraphs, rooted connectivity, directed Steiner tree.

MSC codes: 05C20, 05C21, 05C40, 05C85, 68W25, 90C27, 90C35

Abbreviated title: APPROXIMATION ALGORITHMS FOR k-(S, T) CONNECTIVITY

*Dept. of Comb. & Opt., U. Waterloo, Waterloo ON Canada N2L 3G1. Email: jcheriyan@uwaterloo.ca

†Dept. of Computer Science, McGill University, Montreal QC Canada Email: blaekh@cs.mcgill.ca

1 Introduction

1.1 The model of k-(S, T) connectivity

We introduce a model for NP-hard problems pertaining to the connectivity of graphs. One of the well-known NP-hard problems is to find a minimum-cost strongly connected spanning subgraph of a directed network. In the *minimum-cost* k-(S,T) *connected digraph* (abbreviated, k-(S,T) connectivity) problem we are given an integer $k \geq 0$, a directed graph $G = (V, E_0 \cup E)$ with positive costs on the edges in E, and two subsets S, T of V; we may assume that each edge in E_0 has zero cost. We use n to denote the number of vertices. The goal is to find a subset of edges $\widehat{E} \subseteq E$ of minimum cost such that the subgraph $(V, E_0 \cup \widehat{E})$ has k edge-disjoint directed paths (abbreviated, dipaths) from each vertex in S to each vertex in T. When $k = 1$ and $S = T = V$, we get the minimum-cost strongly connected spanning subgraph (abbreviated, SCSS) problem.

We call E the set of *augmenting edges*, and we call $G_0 = (V, E_0)$ the *initial digraph*. The vertices in $V - (S \cup T)$ are called *optional*. The initial digraph $G_0 = (V, E_0)$ can be arbitrary. Throughout, we use n and m to denote the number of vertices and the number of edges, respectively. We use opt to denote the cost of an optimal solution, and we use E^* to denote the set of edges in an (fixed, arbitrary) optimal solution. When $k = 1$ we drop the connectivity parameter k and refer to our problem as (S, T) connectivity.

Our model of k-(S,T) connectivity, in its full generality, is at least as hard for approximation as the label-cover problem, even when the connectivity parameter k is one. There is a simple reduction from the directed Steiner network (a.k.a. directed Steiner forest) problem, see Proposition 1, and it is well known that the latter problem is at least as hard as the label-cover problem, see [1], [34, Corollary 16.39].

Rather than focusing on this general version of the problem, most of our results focus on a specialized version that we call the standard version, where every edge of positive cost has its tail in S and its head in T. This version of the problem captures NP-hard problems such as the minimum-cost k-edge connected spanning subgraph (abbreviated, k-ECSS) problem and the minimum-cost k-vertex connected spanning subgraph (abbreviated, k-VCSS) problem, which have been extensively studied in the area of approximation algorithms for almost two decades yet there are significant problems left open. Moreover, this version of the problem generalizes the special case of the directed subset k-connectivity problem where every edge of positive cost has both endvertices in the set of terminals. We call it the standard version because a still further specialization of it was introduced and studied by Frank and Jordan more than fifteen years ago [15]. Part of their motivation was to extend their famous min-max theorem giving an optimal characterization for the vertex-connectivity augmentation problem on directed graphs to the more general setting of the k-(S,T) connectivity augmentation problem, where every edge from S to T is present and has unit cost. Thus the model introduced and studied in [15] is polynomial-time solvable; subsequently, improved algorithms were presented by [33]. Moreover, [15] proves min-max results for some of these problems. To the best of our knowledge, the minimum-cost version of the k-(S,T) connectivity augmentation model of [15] has not been studied before. We mention that all of the problems studied in this paper are on digraphs, thus k-VCSS and k-ECSS denote problems on digraphs; some of the literature studies similar problems on undirected graphs; it can be seen that an approximation guarantee of ρ for the k-VCSS problem on digraphs implies an approximation guarantee of 2ρ for the k-VCSS problem on undirected graphs; the same holds for the k-ECSS problem.

One of our results is an approximation algorithm with a guarantee of $O((\log k)(\log n))$ for the standard version of the k-(S,T) connectivity problem. Observe that any approximation guarantee for the min-cost k-(S,T) connected digraph problem implies the same approximation guarantee for the k-VCSS problem.

We do not improve on the best known approximation guarantee for the latter problem, which is $O(\log k \cdot \log(\frac{n}{n-k}))$ [26]. Our algorithm is based on an algorithmic paradigm that we call the *halo-set method*; it was used previously in [9], building on previous work by [24], and others. Recently, Nutov [28, Theorem 1.1] applied it to the problem of covering a crossing biset family by a set of edges of minimum cost, to get an $O(\log n)$ approximation algorithm.

An immediate question is whether our approximation guarantee for k-(S,T) connectivity is optimal, or (more realistically) whether the approximation guarantee can be improved substantially. For $k = O(1)$, note that our approximation guarantee is $O(\log n)$. It is not clear whether there exists a logarithmic (in n) hardness threshold for $k = O(1)$. For $k = 1$, we give a simple 2-approximation algorithm that generalizes a well-known 2-approximation algorithm of [17, 21] for the SCSS problem. But already for $k = 2$, there are substantial difficulties. We give a 3-approximation algorithm in Section 4; the algorithm is simple, but the analysis is nontrivial. We could not find any simple way to achieve an approximation guarantee of $O(1)$ for the 2-(S,T) connectivity problem.

Besides the standard version, we study another version that is intermediate between the standard version and the problem in its full generality (which is label-cover hard). In the relaxed version of the (S,T) connectivity problem, each edge of positive cost has its head in T but there is no restriction on the tail. We study the relaxed version with the connectivity parameter k fixed at one, and observe that this version is at least as hard to approximate as the directed Steiner tree problem. The latter problem has a hardness threshold of $\Omega(\log^{2-\epsilon} n)$ assuming that NP is not contained in ZPTIME$(n^{\text{polylog } n})$ [19]. Let $\alpha(n)$ denote the best approximation guarantee available for the directed Steiner tree problem; the results of [3, 20] show that an approximation guarantee of $O(\log^3 n)$ can be achieved in time $(n^{O(\log n)})$. We give an algorithm that achieves an approximation guarantee of $\alpha(n) + 1$ for the relaxed (S,T) connectivity problem. The key to the analysis is a structural result that decomposes any feasible solution into a set of junction trees that are disjoint on the vertices of T; in fact, each vertex of T appears in exactly one of these junction trees. (See [7, 4] for other applications of junction trees.) Our algorithm and analysis specialize to the case when the digraph is acyclic on T, meaning that there exists no dicycle that contains two distinct vertices of T. In this setting, we show that the relaxed (S,T) connectivity problem is at least as hard to approximate as the set covering problem, and we prove that our algorithm achieves a matching approximation guarantee of $O(\log n)$.

In brief, the model of k-(S,T) connectivity captures several of the keystone problems in the area of approximation algorithms for network design, such as the SCSS problem and its extensions to k-edge connectivity, the k-VCSS problem (Section 5.1), the directed Steiner-tree problem (Proposition 1), the set covering problem (Theorem 5), and the directed Steiner network problem (Proposition 1). Despite this versatility, the model is amenable to simple algorithmic schemes that give state-of-the-art approximation guarantees; meaning that the approximation guarantees are almost as good as those for the well-studied studied special cases.

Figure 1 summarizes the model via the three versions of k-(S,T) connectivity by illustrating some of the key NP-hard problems in network design captured by it.

Many of the results of this paper were first presented in the second author's thesis [25]. We mention that many other types/models of connectivity problems have been studied from the perspective of approximation algorithms [6, 11, 27], etc., but we restrict most of our discussion to the literature that connects directly to our model.

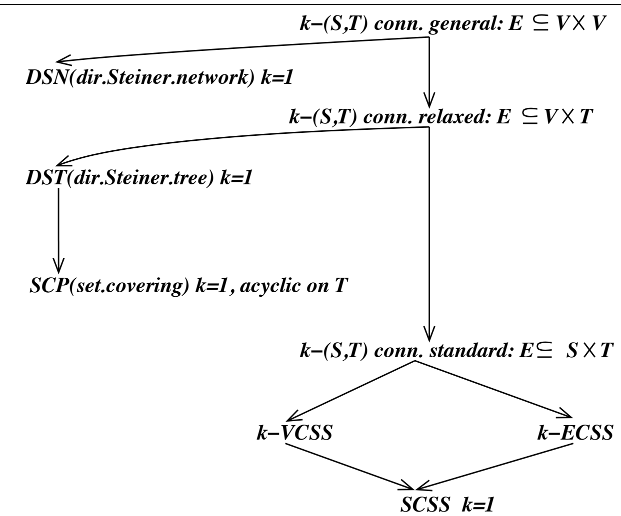


Figure 1: An illustration of the model of k-(S, T) connectivity, showing some of the key NP-hard problems in network design captured by it.

1.2 Frank's algorithm for rooted connectivity

Frank (see [16, 12, 13, 14]) gave polynomial-time algorithms and min-max theorems for finding a min-cost "rooted out-subgraph". More precisely, Frank's results focus on the special case of the min-cost (S, T) connectivity problem where S consists of a single vertex, called the root, and every augmenting edge has its head in T, that is, $(v, w) \in E \Longrightarrow w \in T$. Note that the restriction on E is critical; without this restriction, the directed Steiner tree problem would be a special case of this problem.

We stress that Frank's results (see Theorem 9) immediately give an approximation guarantee of $\min\{|S|, |T|\}$ for the k-(S, T) connectivity problem. The main point of the results in our paper is to obtain substantial improvements on this approximation guarantee. (We have no results of our own on the problems addressed by Frank.)

1.3 Summary of results on k-(S, T) connectivity

This subsection summarizes our results in the model of k-(S, T) connectivity, see Sections 3–6. These results are proved under the assumption that the sets S and T are disjoint; this is without loss of generality, see Proposition 7.

Proposition 1. *Consider the k-(S, T) connectivity problem with the connectivity parameter k equal to one. The hardness of approximation of the problem depends on the version of the problem (and thus on the restrictions on the augmenting edges).*

(1) The standard (S, T) connectivity problem is at least as hard as the SCSS problem.

(2) The relaxed (S, T) connectivity problem is at least as hard as the directed Steiner tree problem.

(3) The (S, T) connectivity problem (without any restrictions on the augmenting edges) is at least as hard as the directed Steiner network problem.

We have (almost) matching approximation guarantees for the first two versions.

The next result gives a 2-approximation algorithm for the standard (S, T) connectivity problem, and the details are given in Section 3. Note that no approximation guarantee better than 2 is known for the SCSS problem which is a special case of our problem (standard (S, T) connectivity), although the former problem (SCSS) has been studied for almost two decades.

Theorem 2. *Consider the standard version of the (S, T) connectivity problem; thus $k = 1$. There is a 2-approximation algorithm that runs in polynomial time.*

The next result addresses the 2-(S, T) connectivity problem; the details are in Section 4. For the special case of the 2-VCSS problem, the best approximation guarantee known is 3 [23].

Theorem 3. *Consider the standard version of the 2-(S, T) connectivity problem. There is a 3-approximation algorithm that runs in polynomial time.*

The next result addresses the standard version of the k-(S, T) connectivity problem; the details are in Sections 5. For the special case of the k-VCSS problem, the best approximation guarantee known is $O(\log k \cdot \log(\frac{n}{n-k}))$ [26].

Theorem 4. *There is a polynomial-time approximation algorithm for the standard version of the min-cost k-(S, T) connected digraph problem that achieves a guarantee of $O(\log k \cdot \log n)$.*

The next result gives an almost tight approximation guarantee for the relaxed version of the (S, T) connectivity problem, where each augmenting edge has its head in T but the tail is unrestricted. The guarantee is tight up to an additive term of one. Moreover, we have some results on the relaxed (S, T) connectivity problem on a restricted class of digraphs. These results are proved in Section 6.

Theorem 5. *Consider the relaxed k-(S, T) connectivity problem with the connectivity parameter k equal to one.*

(1) There exists a $(\alpha(n) + 1)$-approximation algorithm, where $\alpha(n)$ denotes the (best available) approximation guarantee for the directed Steiner tree problem. In particular, there is an $O(\log^3 n)$-approximation algorithm that runs in quasi-polynomial time.

(2) Consider the special case of the problem where the given digraph G is acyclic on T. This problem is at least as hard as the set covering problem. Moreover, there is an $O(\log |S|)$-approximation algorithm that runs in polynomial time.

2 Preliminaries

Most of our notation and terms are standard, see e.g., Schrijver [31]. When we say that two sets S_1, S_2 intersect (or, are intersecting), then we mean that $S_1 \cap S_2$ is nonempty. We call a directed graph a *digraph*, and call a directed path a *dipath*. Suppose that $G = (V, E)$ is a graph or digraph that is an input for our problem instance; then we use n to denote $|V|$, m to denote $|E|$, and, when stating running times, we assume $m = \Omega(n)$. By an *edge* we mean an arc (directed edge) of a digraph, as well as an undirected edge of a graph. For a set of nodes U and a set of edges F of a digraph, $\delta_F^{out}(U)$ denotes the set of edges in F with tail in U and head not in U, thus, $\delta_F^{out}(U) = \{(v, w) \in F \,:\, v \in U,\ w \in V - U\}$, and $d_F^{out}(U)$ denotes the size of this set, $|\delta_F^{out}(U)|$; $\delta_F^{in}(U)$ and $d_F^{in}(U)$ are defined similarly. Given two vertices s, t, an s, t *dipath* means a dipath with start-vertex s and end-vertex t. Given a digraph $G = (V, E)$ and two sets of vertices $S, T \subseteq V$, we use (S, T) *connectivity* to mean the minimum over all pairs $s \in S, t \in T$ of the maximum number of edge-disjoint s, t dipaths; by Menger's theorem, this equals $\min_{s \in S, t \in T}\{|F| \,:\, F \subseteq E,\ G{-}F$ has no s, t dipath$\}$. We say that the digraph is (S, T) connected (or, has (S, T) connectivity of one) if there exists an s, t dipath for each pair of vertices $s \in S, t \in T$. Similarly, we say that the digraph is k-(S, T) connected if it has k edge-disjoint s, t dipaths for each pair $s \in S, t \in T$. We assume that the sets S and T are disjoint in Sections 3–6; this is without loss of generality, see Proposition 7.

By a T, S *dipath*, we mean a dipath that has its start-vertex in T and its end-vertex in S.

Fact 6. *In the relaxed version of (S, T) connectivity (where all augmenting edges have heads in T), the input digraph G has a T, S dipath iff the initial digraph G_0 has a T, S dipath. Moreover, G has ν edge-disjoint T, S dipaths iff G_0 has ν edge-disjoint T, S dipaths.*

Proof. Consider the first part. Let P be any T, S dipath of G. If P has no augmenting edges, then it is in G_0. Otherwise, consider the suffix of P between the last augmenting edge and the end-vertex of P. This subpath has a start vertex in T, an end vertex in S, and no augmenting edges, so it is a T, S dipath of G_0.

The second part follows in a similar way. $\square$

The standard version (where all augmenting edges have tails in S and heads in T) is a special case of the relaxed version, hence, G has a T, S dipath iff the initial digraph G_0 has a T, S dipath.

A digraph $G = (V, E)$ with $T \subseteq V$ is called *acyclic on* T if there is no dicycle in G that contains two distinct vertices of T (by a dicycle we mean a connected subgraph having the same in-degree and out-degree at every vertex). Observe that if G contains both a t, t' dipath and a t', t dipath for $t, t' \in T$, then the union of these two dipaths is a dicycle. Another way to view this is via the reachability digraph on T; this is an auxiliary digraph with vertex set T, and for $t, t' \in T$, $t \neq t'$, it has an edge (t, t') iff G has a t, t' dipath; observe that G is acyclic on T iff the reachability digraph on T has no dicycles.

Let r be a vertex. An *in-tree (or in-branching) J^{in} rooted at* r is a minimal digraph (w.r.t. the edge set) that has a v, r dipath for each vertex $v \in V(J^{in})$. An *out-tree (or out-branching) J^{out} rooted at* r is a minimal digraph (w.r.t. the edge set) that has an r, v dipath for each vertex $v \in V(J^{out})$. Edmonds [8] gave a polynomial-time algorithm that finds a minimum cost in-branching (respectively, out-branching), see also [13, 14].

A *directed Steiner tree* rooted at r is a digraph that has an r, t dipath for every vertex $t \in T$, where $T \subseteq V$ is a given set of *terminal* vertices; the digraph may contain vertices of $V - \{r\} - T$; these are called the Steiner vertices or the optional vertices. In the *directed Steiner tree* problem, we are given a digraph $G = (V, E)$, nonnegative costs on the edges, $r \in V$, and $T \subseteq V$; the goal is to find a directed Steiner tree of minimum cost. The problem has a hardness threshold of $\Omega(\log^{2-\epsilon} n)$ assuming that NP is not contained in $\mathrm{ZPTIME}(n^{\mathrm{polylog}\, n})$, [19]. Charikar et al. [3] gave an $O(\log^3 n)$-approximation algorithm for the directed Steiner tree problem that runs in quasi-polynomial time, also see [20]. More precisely, this algorithm achieves an approximation guarantee of $O(\ell^3 |T|^{1/\ell})$ in a running time of $n^{O(\ell)}$, where ℓ is a positive number; thus, fixing $\ell = \log |T|$ gives an approximation guarantee of $O(\log^3 |T|)$ in quasi-polynomial time.

In the *directed Steiner network* problem, also known as the directed Steiner forest problem, we are given a digraph $G = (V, E)$, nonnegative costs on the edges, and a set of requirement pairs $D \subseteq V \times V$; the goal is to find a subgraph (V, F) of minimum cost that contains an s_i, t_i dipath for each requirement pair $(s_i, t_i) \in D$. This problem is at least as hard for approximation as the label-cover problem, see [1], [34, Corollary 16.39]; in particular, assuming that NP is not contained in DTIME($n^{\text{polylog } n}$), the problem has a hardness threshold of $2^{\log^{(1-\epsilon)} n}$, for any fixed $\epsilon > 0$.

In the directed minimum-cost *k-edge connected spanning subgraph* problem we are given a digraph $G = (V, E)$, and nonnegative costs on the edges; the goal is to find a subgraph (V, F) of minimum cost that contains k edge-disjoint s, t dipaths for each ordered pair of vertices s, t.

In the directed *subset k-connected subgraph* problem, we are given a digraph $G = (V, E)$, nonnegative costs on the edges, and a set of terminals $T \subseteq V$; the goal is to find a subgraph (V, F) of minimum cost that contains k openly disjoint s, t dipaths for each ordered pair of vertices $s, t \in T$. The directed minimum-cost *k-vertex connected spanning subgraph* problem is the special case where $T = V$.

In the *set covering* problem we are given a ground-set U of so-called points, subsets $S_1, \ldots, S_q$ of U, and a nonnegative cost for each subset S_j, $j = 1, \ldots, q$; the goal is to cover U by picking a family of subsets from $S_1, \ldots, S_q$ of minimum cost, that is, each point of U should be in at least one of the picked subsets. A greedy algorithm achieves an approximation guarantee of $O(\log \max_{j=1}^{q} |S_j|)$, and there exists a constant c such that improving on the approximation guarantee of $(c \log |U|)$ in polynomial time would imply that $\mathrm{P} = \mathrm{NP}$, see [2, 30, 10].

2.1 Basic results

This subsection has some basic results on the k-(S, T) connectivity problem.

Proposition 7. *There is a polynomial-time reduction from instances of the (S, T) connectivity problem (with the connectivity parameter k equal to one) where $S \cap T \neq \emptyset$ to instances such that $S \cap T = \emptyset$ that preserves the feasibility and the cost of candidate solutions.*

Proof. For each vertex $v \in S \cap T$, we split v into two vertices v^+ and v^-, and join them by a pair of new edges (v^-, v^+) and (v^+, v^-) with zero cost. For each old edge having v as tail (respectively, head), we change the tail (respectively, head) to v^+ (respectively, v^-). Finally, for each vertex $v \in S \cap T$, we replace v by v^+ in S and we replace v by v^- in T. It can be seen that this preserves any restrictions on the augmenting edges, that is, an edge has its head in T (respectively, has its tail in S) iff the old edge corresponding to it has its head in T (respectively, has its tail in S).

We can map any dipath in the original instance to a dipath in the transformed one by replacing a vertex $v \in S \cap T$ by a subpath v^-, v^+. Conversely, we can map any dipath in the transformed instance to a dipath in the original one by replacing subpaths v^-, v^+ or v^+, v^- (or v^+ or v^-) by a single vertex v. Edges of the form (v^+, v^-) guarantee that S, T dipaths of the form v (with no edges) in the original instance map to S, T dipaths of the form v^+, v^- in the transformed instance. Hence, a subgraph (candidate solution) of the original instance is feasible iff the corresponding subgraph of the transformed instance is feasible. Moreover, optimal solutions of both instances have the same cost. □

Remark 8. There is a similar reduction for the k-(S, T)-connectivity problem that reduces instances with $S \cap T \neq \emptyset$ to instances such that $S \cap T = \emptyset$: the only difference is that we keep k parallel copies of the edge (v^-, v^+), as well as k parallel copies of the edge (v^+, v^-).

The proofs of our hardness-of-approximation results on different versions of the (S, T) connectivity problem are given below; the results are stated in Proposition 1 in Section 1.3.

Proof of Proposition 1. We will describe the hardness construction of each version of the problem. We recall that an instance of the (S, T) connectivity problem consists of a digraph $G = (V, E_0 \cup E)$, sets of vertices S and T, and a positive cost $c(e)$ on each augmenting edge $e \in E$.

The standard (S, T) connectivity problem: The reduction from the SCSS problem to the standard (S, T) connectivity problem is straightforward. In fact, the SCSS problem is a special case of the standard (S, T) connectivity problem where $S = T = V$. It is clear that the restriction on heads and tails of augmenting edges holds because $S = T = V$. Moreover, By Proposition 7, we can transform this instance to an instance such that $S \cap T = \emptyset$.

The relaxed (S, T) connectivity problem: The reduction from the directed Steiner tree problem is as follows. The given instance of the in-directed Steiner tree problem consists of a digraph $G' = (V', E')$ with non-negative cost on edges, a root vertex $r \in V'$ and a set of terminals $S' \subseteq V'$. We may assume that $r \notin S'$. Moreover, we may assume that each terminal $s \in S'$ is incident to a unique edge which is outgoing from s and has zero-cost. Otherwise, we can replace each terminal $s \in S'$ by a dummy terminal s^+ and attach s^+ to s by a zero-cost edge (s^+, s). Observe that the reduction does not increase the cost or violate the feasibility of an optimal solution. We construct the digraph G for the instance of the relaxed (S, T) connectivity problem by starting with G', and then adding auxiliary edges with zero-cost from the root vertex r to all non-terminal vertices. Thus, the digraph is $G = (V, E_0 \cup E)$, where $V = V'$ and $E_0 \cup E = E' \cup \{(r, v) : v \in V' - S'\}$. We define S to be the set of terminals S' and T to be the set of non-terminal vertices, that is, $S = S'$ and $T = V - S$. Note that T also includes the root vertex r. We define the set of edges E_0 of the initial digraph to be the set of all zero-cost edges including the auxiliary ones. We define the set of augmenting edges E to be the set of all positive-cost edges. The construction is valid for the relaxed (S, T) connectivity problem because all positive-cost edges have heads in T, the set of non-terminal vertices. It can be seen that the reduction is approximation-preserving. The reduction is illustrated in Figure 2.

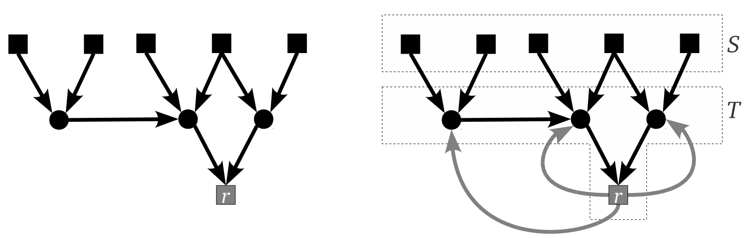


Figure 2: The reduction from an instance of the directed Steiner tree problem to an instance of the relaxed (S, T) connectivity problem. The left figure shows the instance of the former problem. The squares denote terminals, and the circles denote Steiner vertices; r is the root vertex. The right figure shows the instance of the relaxed (S, T) connectivity problem. The black lines denote positive cost edges, and the grey lines denote zero-cost edges.

The (general) (S, T) connectivity problem: The reduction from the directed Steiner network problem is as follows. The given instance of the directed Steiner network problem consists of a digraph $G' = (V', E')$ with non-negative cost on the edges, and a set of requirement pairs $D \subseteq V' \times V'$. We may assume that there exist a set of sources $S \subseteq V'$ and a set of sinks $T \subseteq V'$ such that $D \subseteq S \times T$. Moreover, we may assume that each source $s \in S$ is incident to one outgoing edge but incident to no incoming edges. Similarly, we may assume that each sink $t \in T$ is incident to one incoming edge but incident to no outgoing edges. The reduction can be done similarly to that of the directed Steiner tree problem. For each source $s \in S$, we add a dummy vertex s^+ and attach it to s by a zero-cost edge (s^+, s). Likewise, for each sink $t \in T$, we add a dummy vertex t^- and attach it to t by a zero-cost edge (t, t^-). We then replace the requirement pair (s,t) by (s^+, t^-) for all $(s,t) \in D$. Since the dummy sources and sinks are attached to the original ones by zero-cost edges, the reduction does not increase the cost or violate the feasibility. Note that each source and sink may occur in more than one requirement pair, e.g., we may have both (s, t_1) and (s, t_2) in D. We construct the digraph G for the instance of the (general) (S,T) connectivity problem by starting with G', and then adding some auxiliary edges with zero cost. We define S and T to be the set of sources and sinks, respectively. For all ordered pairs (s,t) with $s \in S$ and $t \in T$, if $(s,t) \notin D$, then we add an auxiliary edge (s,t) to G with zero cost. In other words, we pad the digraph with auxiliary edges to handle the new requirement pairs implicit in the (general) (S,T) connectivity problem. The set of edges E_0 is defined to be the set of all zero-cost edges including the auxiliary ones. The set of augmenting edges E is defined to be the set of positive-cost edges. The construction is valid for the (general) (S,T) connectivity problem because there is no restriction on augmenting edges. It can be seen that the reduction is approximation-preserving. $\square$

2.2 Frank's result on rooted connectivity

Recall that a digraph $G' = (V', E')$ is said to be k-(r,T) connected, where $k \geq 0$ is an integer, $r \in V'$ and $T \subseteq V'$, if it has k edge-disjoint r,t dipaths, for each $t \in T$. Below, we state a result of Frank on a version of the min-cost k-(r,T) connectivity problem: there is an LP relaxation that is integral, i.e., the problem can be captured as a linear programming problem. Our algorithmic results in Section 5 rely on Frank's results on rooted k-connectivity. Also, our algorithms in Section 4 apply Frank's results with $k = 2$.

Theorem 9 (Frank 2009 Theorems 4.4,5.9[14])**.** *Given a digraph $G = (V, E_0 \cup E)$, a vertex $r \in V$, a set of vertices $T \subseteq V - \{r\}$ that contains the head of every edge in E, and positive costs on the edges in E, there is a polynomial-time algorithm for finding a set of edges $F \subseteq E$ of minimum cost such that the subgraph $(V, E_0 \cup F)$ is k-(r,T) connected. Moreover, the optimal cost equals the optimal value of a linear-programming (LP) relaxation.*

It is easily seen that Frank's result applies also for the min-cost "rooted in-subgraph" problem, by replacing each edge (v,w) by its reverse edge (w,v) and making appropriate changes for T and E; that is, we have a set of vertices $S \subseteq V - \{r\}$ that contains the tail of every edge of positive cost, and the goal is to find a subgraph of minimum cost that is k-(S,r) connected.

2.3 Reducing algorithm for digraphs

This subsection presents a simple algorithm that, given a digraph and a set of its vertices Z, finds a minimal subset of Z that preserves some reachability properties; precise statements are given in the next result. We apply this algorithm in Sections 3 and 4.

Proposition 10. *Let $H' = (V', E')$ be a digraph, and let $Z \subseteq V'$ be a set of vertices. Then there is a linear-time algorithm to find a subset Y of Z such that*

(1) for each vertex $v \in Z$, there is a vertex $y \in Y$ such that H' has a v,y dipath,

(2) H' has no dipath from any vertex of Y to another vertex of Y, and

(3) if H' has a dipath from a vertex $y \in Y$ to a vertex $v \in Z$, then y and v are in the same strongly-connected component of H'.

Proof. We apply the following method to output the vertices of Y sequentially. This method can be implemented to run in linear time.

We contract the strongly-connected components of H' into single vertices, to obtain an acyclic digraph H''. Observe that Y can have at most one vertex from each strongly-connected component of H'. Thus we transform the problem from H' to the acyclic digraph H'' in an obvious way. (A vertex v of H'' is in Z iff the strongly-connected component of H' associated with v contains a vertex of Z; similarly, a set Y of H'' can be mapped to a set Y of H' of the same size.) Then we assign a topological numbering to the vertices of H''.

We start with $X = Z$. We output the vertex y of H'' that is in X and has the highest topological number; thus this vertex is placed in Y. Then we find the vertices of X that have dipaths to y in H''; we remove all these vertices (including y) from X. We repeat this step until X becomes empty.

This method can be implemented to run in time $O(|V'| + |E'|)$. At the start, we can construct H'', a topological numbering of $V(H'')$, and a labeling of $V(H'')$ by $X = Z$ in linear time, see [31]. We have an outer loop that scans the vertices of H'' according to the topological numbering (highest to lowest); whenever the outer loop finds a vertex y of X, then y is placed in Y, and then we execute an inner loop that visits all vertices that have dipaths to y in H'', and then we remove all these vertices from H'' and X. Finally, the mapping of Z from H' to H'', and the mapping of Y from H'' to H' can be computed in linear time.

To verify the correctness, note that the initial set X satisfies

($*$) for each vertex $v \in Z - X$ there is a vertex $y \in Y$ such that H' has a v, y dipath.

Whenever we add a vertex y to Y, we remove from X all vertices v that can reach y. Clearly, this preserves ($*$). At termination, $X = \emptyset$ and ($*$) holds, hence, (1) holds. Consider (2): whenever we add a vertex y to Y, there is no dipath from y to vertices already in Y, and there is no dipath from y to vertices in the current set X; hence, (2) holds. Consider (3): whenever we add a vertex y to Y, then y has the highest topological number among the vertices in the current X; this implies that (3) holds. (If (3) fails, then there is a vertex $y \in Y$ and a vertex $v \in Z$ such that v and y are in different strongly-connected components of H', and there is a y, v dipath in H'. Thus, y and v are associated with distinct vertices of H'', call them y'' and v'', and H'' has a dipath from y'' to v''. This is not possible because when we add y to Y, then either $v \in X$ or $v \notin X$; in the first case, v has a higher topological number than y, and in the second case, y and v would have been removed from X at the same step.) □

3 A 2-approximation algorithm for standard (S, T) connectivity

This section has our 2-approximation algorithm for the standard version of the (S, T) connectivity problem, that is, the problem of finding an (S, T) connected digraph of minimum cost, assuming that each augmenting edge has its tail in S and its head in T.

This problem is a generalization of the SCSS problem. We sketch a well-known 2-approximation algorithm for the latter problem, see [17]: Let opt denote the cost of an optimal solution. The algorithm picks any vertex to be the root vertex r, and then computes a min-cost out-branching (V, F^{out}) with root r; similarly, the algorithm computes a min-cost in-branching (V, F^{out}) with root r; then, the algorithm outputs

$(V, F^{out} \cup F^{in})$. It can be seen that the solution is strongly-connected. Moreover, it can be seen that the cost of the solution is $\leq 2\mathrm{opt}$.

Recall that a T, S dipath is a dipath that has its start-vertex in T and its end-vertex in S; moreover, G has a T, S dipath iff the initial digraph G_0 has one.

We consider two cases:

1. G has a T, S dipath.
2. G has no T, S dipath.

We give a 2-approximation algorithm for the first case in Section 3.1, by designing a simple extension of the above 2-approximation algorithm for the SCSS problem. We handle the second case in Section 3.2 by giving a polynomial-time algorithm that solves it optimally.

To solve the standard (S, T) connectivity problem, we first run a depth first search algorithm to find a T, S dipath, if one exists. If there is no such dipath, then we run the algorithm given in Section 3.2. Otherwise, we run the 2-approximation algorithm given in Section 3.1. Combining these two cases, we get a 2-approximation algorithm for the problem. This proves Theorem 2 in Section 1.3.

3.1 A 2-approximation algorithm for the case when a T, S dipath exists

First, suppose that there exists a T, S dipath, call it $\widehat{P}$, with start-vertex $\widehat{t} \in T$ and end-vertex $\widehat{s} \in S$. Then, we apply Frank's algorithm (see Theorem 9) for min-cost rooted in-subgraphs to our digraph $G = (V, E_0 \cup E)$ with root $\widehat{t}$, to find a min-cost set of edges F^{in} such that the rooted in-subgraph $(V, E_0 \cup F^{in})$ is $(S, \widehat{t})$ connected, that is, the subgraph contains an $s, \widehat{t}$ dipath for each vertex $s \in S$. Next, we apply Frank's algorithm (see Theorem 9) for min-cost rooted out-subgraphs to our digraph $G = (V, E_0 \cup E)$, but now we take the root to be $\widehat{s}$ and we find a min-cost set of edges F^{out} such that the "out-subgraph" $(V, E_0 \cup F^{out})$ is $(\widehat{s}, T)$ connected, that is, the subgraph contains an $\widehat{s}, t$ dipath for each vertex $t \in T$. The algorithm outputs $F^{in} \cup F^{out}$ as its solution. Below, we show that the algorithm is correct, that is, the subgraph $\widehat{G} = (V, E_0 \cup F^{in} \cup F^{out})$ is (S, T) connected, and it achieves an approximation guarantee of 2, that is, $c(F^{in} \cup F^{out}) \leq 2\mathrm{opt}$.

Proposition 11. *Suppose that the digraph G has a T, S dipath. Then the above algorithm runs in polynomial time and finds a feasible solution of cost $\leq 2\mathrm{opt}$.*

Proof. First, we show that the output is correct, that is, the digraph $\widehat{G}$ returned by the algorithm is (S, T) connected. Consider any pair of vertices s, t where $s \in S$ and $t \in T$. Observe that $\widehat{G}$ is $(S, \widehat{t})$ connected; similarly, $\widehat{G}$ is $(\widehat{s}, T)$ connected. Hence, $\widehat{G}$ has an s, t dipath of the form

$$s \to \ldots (F^{in}) \ldots \to \widehat{t} \to \ldots (\widehat{P}) \ldots \to \widehat{s} \to \ldots (F^{out}) \ldots \to t.$$

Therefore, $\widehat{G}$ is (S, T) connected.

To see that the algorithm achieves an approximation guarantee of 2, note that a set of edges that is feasible to the (S, T) connectivity problem is also feasible to each of the two rooted subproblems solved by Frank's algorithm, hence, the cost of a feasible solution to each rooted subproblem is $\leq \mathrm{opt}$. Therefore, $\widehat{G}$ has cost $\leq 2\mathrm{opt}$.

The algorithm runs in polynomial time because a T, S dipath of G_0 can be found in linear time (if it exists), and each of the two applications of Frank's algorithm runs in polynomial time (see Theorem 9). This completes the proof. $\square$

Figure 3 illustrates the working of this algorithm.

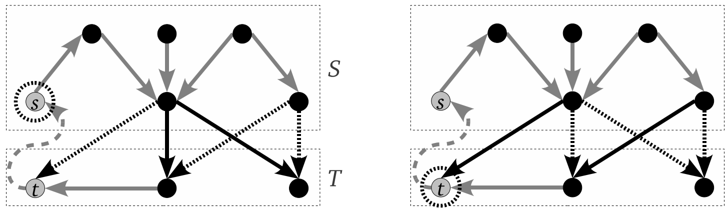


Figure 3: The figure illustrates the working of our 2-approximation algorithm for the standard (S,T) connectivity problem on an example that has a T,S dipath. The left figure shows an instance of the rooted out-subgraph problem with root vertex s and set of terminals T. The right figure shows an instance of the rooted in-subgraph problem with a root vertex t and a set of terminals S. The grey lines denote the edges of the initial digraph G_0. The grey dash-lines denote the T,S dipath. The black lines (solid and dash) denote the augmenting edges. The black solid-lines denote the augmenting edges chosen by the algorithm.

3.2 An algorithm for the case of no T,S dipath

Recall the reducing algorithm from Section 2.3, and the conditions (1), (2), and (3) from Proposition 10. We apply that algorithm to the initial digraph G_0 and we take $Z = S$. We take the output $Y \subseteq Z$ to be $\widetilde{S} \subseteq S$. Then conditions (1), (2), and (3) apply to $\widetilde{S}$ and S. (Thus (1) for each $s \in S$ there is an $s' \in \widetilde{S}$ such that G_0 has an s,s' dipath; (2) G_0 has no dipath between two vertices of $\widetilde{S}$, and (3) if G_0 has a dipath from $s' \in \widetilde{S}$ to $s \in S$, then both s,s' are in the same strongly-connected component of G_0.) After that, for each vertex $s_i \in \widetilde{S}$, we apply Frank's "rooted out-subgraph" algorithm to the original digraph, taking the root to be s_i, and we compute an augmenting edge set $F^{out}(s_i)$ of minimum cost such that $(V, E_0 \cup F^{out}(s_i))$ is (s_i,T) connected.

Proposition 12. *Suppose that the digraph G has no T,S dipath. Then the above algorithm finds an optimal solution to the (S,T) connectivity problem. The algorithm runs in time $O(|\widetilde{S}|m(n+m)) = O(nm^2)$.*

Proof. Let $\widehat{G}$ denote the digraph returned by the algorithm, that is, $\widehat{G} = (V, E_0 \bigcup_{s_i \in \widetilde{S}} F^{out}(s_i))$. First, we show that $\widehat{G}$ is (S,T) connected. Consider any vertex $s \in S$. Then the set $\widetilde{S}$ found by the reducing algorithm has a vertex s' such that G_0 has a dipath from s to s', by (1) in Proposition 10. Moreover, each vertex in $\widetilde{S}$ has been chosen as the root vertex for an application of Frank's algorithm, hence, G_0 together with the augmenting edge set added via Frank's algorithm contains an s',t' dipath, for all $t' \in T$. Thus, $\widehat{G}$ has an s,t' dipath, for all $t' \in T$.

Next, we show that the cost of $\widehat{G}$ is $\leq \mathrm{opt}$. Consider an optimal set of augmenting edges E^*, and an arbitrary vertex $s \in \widetilde{S}$. $G^* = G_0 + E^*$ must have an s,t' dipath from s to each $t' \in T$. Let $E^*(s)$ denote the subset of the augmenting edges in E^* that are used by these dipaths; the tail vertex of each of these augmenting edges must be reachable from s. Then, by Proposition 10, the tail vertices of these augmenting edges are in the same strongly-connected component as s (by (3) in the proposition); moreover, for any other vertex $s' \in \widetilde{S}$, there is no dipath from s' to any vertex in the strongly-connected component of s (by (2) in the proposition). Thus the sets $\{E^*(s) : s \in \widetilde{S}\}$ form a partition of E^*, i.e., each augmenting edge is in at most one of the sets $E^*(s)$. Finally, note that $E^*(s)$ forms a feasible solution for our application of Frank's "rooted out-subgraph" algorithm with root s, hence, the cost of the augmenting edges for this application is $\leq c(E^*(s))$. Summing over the cost of the augmenting edges for all of the applications of Frank's "rooted out-subgraph" algorithm with roots in $\widetilde{S}$, we see that the total cost is $\leq \sum_{s \in \widetilde{S}} c(E^*(s)) \leq \text{opt}$. Observe that this analysis relies on our assumption that the input digraph G has no T, S dipath; since there are no T, S dipaths, the reducing algorithm returns the same output Y for input $Z = S$ on both digraphs G and G_0; in other words, the addition of augmenting edges has no effect on the output of the reducing algorithm.

The bound on the running time follows from the fact that the reducing algorithm runs in linear time per vertex of $\widetilde{S}$, and each application of Frank's algorithm runs in time $O(m(n+m))$. $\square$

4 A 3-approximation algorithm for standard 2-(S,T) connectivity

We say that a digraph is 2-(S,T) *connected* if it contains two edge-disjoint s,t dipaths for every vertex $s \in S$ and every vertex $t \in T$. In this section, we give a 3-approximation algorithm for the standard version of the *minimum-cost* 2-(S,T) *connected digraph* problem, or, in brief, the standard 2-(S,T) *connectivity* problem. Recall that the standard version has $E \subseteq S \times T$, that is, each augmenting edge has its tail in S and its head in T.

Many of the steps of the algorithm and analysis occur in "symmetric pairs," for example, we may apply some procedure to a vertex in S, and then we may apply a similar procedure to a vertex in T. We describe one of these procedure in detail, but not the second one.

4.1 Preliminaries for the approximation algorithm

Let P be a dipath of G. For vertices v_i, v_j of P, we denote by $P(v_i, v_j)$ the subpath of P that starts with v_i and ends with v_j. Similarly, for edges e_i, e_j of P, we denote by $P(e_i, e_j)$ the subpath of P that starts with e_i and ends with e_j.

Recall that a T, S dipath is a dipath whose start-vertex is in T and whose end-vertex is in S. Let ν denote the maximum number of edge-disjoint T, S dipaths of G. By Fact 6, ν is the same for G and for the initial digraph G_0, Usually, we will consider G_0 whenever we discuss T, S dipaths. Our algorithm has three cases depending on whether ν is zero, one, or more than one. The case of $\nu = 1$ appears to be substantially more difficult than the other two cases.

The algorithm starts with the (edge set of the) initial digraph $G_0 = (V, E_0)$, and adds augmenting edges in several steps. We denote the current digraph by $\widehat{G}$ and the current set of augmenting edges by $\widehat{E}$. Thus, $\widehat{G} = (V, E_0 \cup \widehat{E})$, and initially, $\widehat{G} = G_0$, $\widehat{E} = \emptyset$.

4.2 No T, S dipaths

First, suppose that $\nu = 0$, i.e., G (and G_0) have no T, S dipaths. We focus on the initial digraph G_0.

The key subroutine for our algorithm is an extension of Frank's algorithm for the min-cost "rooted out-subgraph" problem. Frank gave polynomial-time algorithms and min-max theorems for the special case of the k-(S,T) connectivity problem where S consists of a single vertex and every augmenting edge has its head in T, see Theorem 9, and [13, 14]. Thus, given a root vertex r and assuming $E \subseteq V \times T$, Frank's algorithm computes a minimum-cost set of augmenting edges such that the resulting digraph is k-(r,T) connected.

Recall the reducing algorithm from Section 2.3, and the conditions (1), (2), and (3) from Proposition 10. We apply that algorithm to G_0 and we take $Z = S$. We take the output $Y \subseteq Z$ to be $\widetilde{S} \subseteq S$. Then conditions (1), (2), and (3) apply to $\widetilde{S}$ and S. After that, for each vertex $s_i \in \widetilde{S}$, we apply Frank's "rooted

2-outconnected-subgraph" algorithm to the original digraph, taking the root to be s_i, and we compute an augmenting edge set $F^{out}(s_i)$ of minimum cost such that $(V, E_0 \cup F^{out}(s_i))$ is 2-(s_i, T) connected.

Similarly, we apply the reducing algorithm (in Section 2.3) to the initial digraph G_0, to find a subset $\widetilde{T}$ of T such that conditions (1), (2) and (3) of Proposition 10 hold with $\widetilde{T} = Y$ and $T = Z$. (Formally speaking, we apply the algorithm to the digraph obtained from G_0 by replacing each edge (v, w) by the reverse edge (w, v). Thus (1) for each $t \in T$ there is a $t' \in \widetilde{T}$ such that G_0 has a t', t dipath; (2) G_0 has no dipath between two vertices of $\widetilde{T}$, and (3) if G_0 has a dipath from $t \in T$ to $t' \in \widetilde{T}$, then both t, t' are in the same strongly-connected component of G_0.) Then, for each vertex $t_j \in \widetilde{T}$, we apply Frank's "rooted 2-inconnected-subgraph" algorithm to the original digraph, taking the root to be t_j, and we compute an augmenting edge set $F^{in}(t_j)$ of minimum cost such that $(V, E_0 \cup F^{in}(t_j))$ is 2-(S, t_j) connected.

All these augmenting edge sets are added to the current digraph; thus we have $\widehat{E} = \bigcup_{s_i \in \widetilde{S}} F^{out}(s_i)$ $\bigcup_{t_j \in \widetilde{T}} F^{in}(t_j)$.

Lemma 13. *The current digraph $\widehat{G}$ is 2-(S, T) connected.*

Proof. To see that $\widehat{G}$ is 2-(S, T) connected, consider any cut $(U, V-U)$ with $U \cap S \neq \emptyset$ and $(V-U) \cap T \neq \emptyset$. If any vertex s_i of $\widetilde{S}$ is in U, then the cut has ≥ 2 edges (since $\widehat{G}$ is 2-(s_i, T) connected, $\forall s_i \in \widetilde{S}$). Similarly, if any vertex t_j of $\widetilde{T}$ is in $V-U$, then the cut has ≥ 2 edges. In the remaining case, we have $\widetilde{T} \subseteq U$ and $\widetilde{S} \subseteq V-U$. Let s be a vertex of $U \cap S$ and let t be a vertex of $(V-U) \cap T$; both s, t exist by our choice of U. Moreover, by Proposition 10, there is a dipath P_s from s to some vertex $s' \in \widetilde{S}$ (by (1) in the proposition). Similarly, by Proposition 10, there is a dipath P_t from some vertex $t' \in \widetilde{T}$ to t. The dipaths P_s and P_t have no vertex (or edge) in common; otherwise, their union would contain a T, S dipath. It follows that the cut $(U, V-U)$ has at least two edges, one from P_s and one from P_t. This completes the proof. □

The next lemma shows that the cost of the solution digraph $\widehat{G}$ is $\leq \mathrm{opt}$. The proof is similar to the proof of Proposition 12.

Lemma 14. *The cost of the current digraph $\widehat{G}$ is $\leq 2\mathrm{opt}$.*

Proof. The key point is that the total cost of the edges in $\bigcup_{s \in \widetilde{S}} F^{out}(s)$ is $\leq \mathrm{opt}$; similarly, the total cost of the edges in $\bigcup_{t \in \widetilde{T}} F^{in}(t)$ is $\leq \mathrm{opt}$.

Consider the first claim. Consider an optimal set of augmenting edges E^*, and an arbitrary vertex $s \in \widetilde{S}$; let $E^*(s)$ denote the subset of the augmenting edges in E^* that are used by the two edge-disjoint s, t' dipaths from s to each $t' \in T$. Arguing as in the proof of Proposition 12, it can be seen that the sets $\{E^*(s) : s \in \widetilde{S}\}$ form a partition of E^*. Moreover, $E^*(s)$ forms a feasible solution for our application of Frank's "rooted 2-outconnected-subgraph" algorithm with root s, hence, the cost of the augmenting edges for this application is $\leq c(E^*(s))$. Summing over the cost of the augmenting edges for all of the applications of Frank's algorithm with roots in $\widetilde{S}$, we see that the total cost is $\leq \sum_{s \in \widetilde{S}} c(E^*(s)) \leq \mathrm{opt}$. □

4.3 Two edge-disjoint T, S dipaths

Suppose that G has two edge-disjoint T, S dipaths. We consider G_0, since the maximum number of edge-disjoint T, S dipaths is the same in G and in G_0. We find two edge-disjoint T, S dipaths in G_0, call them P_1 and P_2; this can be done via an application of a maximum s, t flow algorithm to find an integral flow, with T as the set of sources and S as the set of sinks. Let t_1, s_1 be the start-vertex and end-vertex of P_1, and let t_2, s_2 be the start-vertex and end-vertex of P_2.

Then, for $i = 1, 2$, we apply Frank's "rooted 2-outconnected-subgraph" algorithm to the original digraph, taking the root to be s_i, and we compute an augmenting edge set $F^{out}(s_i)$ of minimum cost such that $(V, E_0 \cup F^{out}(s_i))$ is 2-(s_i, T) connected. Clearly, the cost of each of these augmenting edge sets is $\leq \mathrm{opt}$.

Finally, we apply Frank's "rooted 2-inconnected-subgraph" algorithm to compute an augmenting edge set $F^{in}(s_1, s_2)$ of minimum cost such that the digraph $(V, E_0 \cup F^{in}(s_1, s_2))$ has 2 edge-disjoint dipaths from each vertex $s \in S$ to the set $\{s_1, s_2\}$; that is, for each $s \in S$, the digraph $(V, E_0 \cup F^{in}(s_1, s_2))$ has an s, s_1 dipath and an s, s_2 dipath such that these two dipaths are edge disjoint. Formally speaking, we construct an auxiliary digraph by adding a new vertex s^{**} and the new edges (s_1, s^{**}), (s_2, s^{**}), and then we apply Frank's "rooted 2-inconnected-subgraph" algorithm to compute an augmenting edge set $F^{in}(s_1, s_2)$ of minimum cost such that the digraph $(V \cup \{s^{**}\}, E_0 \cup \{(s_1, s^{**}), (s_2, s^{**})\} \cup F^{in}(s_1, s_2))$ is 2-(S, s^{**}) connected.

It can be seen that the cost of $F^{in}(s_1, s_2)$ is $\leq \mathrm{opt}$; to see this, consider an optimal set of augmenting edges E^*, and any vertex $s \in S$; clearly, $(V, E_0 \cup E^*)$ contains two edge-disjoint s, t_1 dipaths, two edge-disjoint s, t_2 dipaths, and two edge-disjoint $\{t_1, t_2\}, \{s_1, s_2\}$ dipaths, hence, $(V, E_0 \cup E^*)$ contains two edge-disjoint $s, \{s_1, s_2\}$ dipaths.

We add all these augmenting edge sets to the current digraph, i.e., we let $\widehat{E} = E_0 \bigcup F^{out}(s_1) \bigcup F^{out}(s_2) \bigcup F^{in}(s_1, s_2)$.

Lemma 15. *The current digraph $\widehat{G} = G_0 + \widehat{E}$ is 2-(S, T) connected, and the cost of $\widehat{E}$ is $\leq 3\mathrm{opt}$.*

Proof. Clearly, $c(\widehat{E}) \leq 3\mathrm{opt}$, since each of the three sets of augmenting edges added to $\widehat{E}$ has cost $\leq \mathrm{opt}$.

To see that $\widehat{G}$ is 2-(S, T) connected, consider any cut $(U, V - U)$ with $U \cap S \neq \emptyset$ and $(V - U) \cap T \neq \emptyset$. If s_1 or s_2 is in U, then the cut has ≥ 2 edges (since $\widehat{G}$ is 2-(s_i, T) connected for $i = 1, 2$). Otherwise, U contains some vertex $s \in S - \{s_1, s_2\}$, and $V - U$ contains both s_1 and s_2; then the cut has ≥ 2 edges (since $\widehat{G}$ contains two edge-disjoint $s, \{s_1, s_2\}$ dipaths, for all $s \in S$). This completes the proof. □

4.4 One (but not two) edge-disjoint T, S dipaths

Recall that the maximum number of edge-disjoint T, S dipaths is the same in G and in G_0. Suppose that G_0 has a T, S dipath, but it does not have two edge-disjoint T, S dipaths; thus $\nu = 1$. Then there exists a (T, S) cut of G_0 of size one. (Such a cut can be found by applying the max-flow min-cut theorem and algorithm, with T as the set of sources and S as the set of sinks.) We define a *cut edge* (of G_0, or equivalently of G) to be an edge whose deletion results in a digraph that has no T, S dipaths. (Observe that an edge e is a cut edge w.r.t. G_0 iff e is a cut edge w.r.t. G.)

We start by finding a T, S dipath of G_0 with the minimum number of edges, call it $\widehat{P}$. We denote the start-vertex and end-vertex of $\widehat{P}$ by $\widehat{t}$ and $\widehat{s}$, respectively. Clearly, every cut edge is contained in $\widehat{P}$. Let $e_1, e_2, \ldots, e_\ell$ denote all the cut edges, listed according to their order of occurrence in $\widehat{P}$.

Lemma 16. *Let P be any T, S dipath. Then all of the cut edges are in P, and their order of occurrence is the same in P and $\widehat{P}$, namely, $e_1, e_2, \ldots, e_\ell$.*

Proof. Clearly, each of the cut edges is in P. For the second part, we argue by contradiction. Let j be the smallest index (possibly, $j = 1$) such that e_j does not occur as the jth cut edge of P. Then, $P - e_j$ contains a dipath starting with e_{j-1} (starting with some $t \in T$, if $j = 1$) and ending with e_q, $q \geq j + 1$; then the union of this dipath with $\widehat{P}$ contains a T, S dipath of $\widehat{G} - e_j$. □

We use t^* to denote the tail vertex of e_1 and s^* to denote the head vertex of e_ℓ. (In general, $t^* \notin T$ and $s^* \notin S$, but it turns out that t^* and s^* have some of the properties of the vertices in T and S, respectively.) The next lemma states that the deletion of a non-cut edge from G_0 cannot disconnect t^* from s^*.

Lemma 17. *Let f be an edge of $\widehat{P}(e_1, e_\ell)$ that is not a cut edge. Then $G_0 - f$ has a dipath from t^* to s^*.*

Proof. Since f is not a cut edge, $G_0 - f$ has a T, S dipath P'. Moreover, P' contains all of the cut edges, and their order of occurrence on P' is $e_1, e_2, \ldots, e_\ell$. Thus $P'(e_1, e_\ell)$ is the required dipath of $G_0 - f$ from t^* to s^*. $\square$

We construct the digraph $\widehat{G}$ by starting with G_0, and then applying preprocessing steps to obtain a digraph that is (S, T) connected such that the deletion of any one non cut edge preserves the (S, T) connectivity; in other words, if the removal of a single edge from $\widehat{G}$ results in a digraph that is not (S, T) connected, then the removed edge must be a cut edge.

The preprocessing steps first apply Frank's "rooted 2-inconnected-subgraph" algorithm with root t^*, to compute an augmenting edge set $F^{in}(t^*)$ of minimum cost such that $(V, E_0 \cup F^{in}(t^*))$ is 2-(S, t^*) connected. Then, we apply Frank's "rooted 2-outconnected-subgraph" algorithm with root s^*, to compute an augmenting edge set $F^{out}(s^*)$ of minimum cost such that $(V, E_0 \cup F^{out}(s^*))$ is 2-(s^*, T) connected. Let $\widehat{G}$ be the digraph $G_0 + F^{in}(t^*) + F^{out}(s^*)$; note that the initial digraph is a subgraph of $\widehat{G}$.

Lemma 18. *The digraph $\widehat{G} = G_0 + F^{in}(t^*) + F^{out}(s^*)$ has cost $\leq 2\mathrm{opt}$, and moreover, it is both 2-(S, t^*) connected and 2-(s^*, T) connected.*

Proof. We claim that the digraph $G^* = G_0 + E^*$ given by an optimal solution E^* to the 2-(S, T) connectivity problem satisfies the requirements of both 2-(S, t^*) connectivity and 2-(s^*, T) connectivity. Consider the first requirement. Informally speaking, G^* is 2-(S, T) connected and G_0 has two edge disjoint dipaths from T to t^*, hence, G^* is 2-(S, t^*) connected; a rigorous proof is given below. Similarly, it can be proved that G^* is 2-(s^*, T) connected. Moreover, it can be seen that the condition required by Frank's "rooted 2-inconnected-subgraph" algorithm applies, that is, every edge of positive cost has its tail at a vertex with positive connectivity requirement because every augmenting edge has its tail vertex in S. Therefore, Frank's algorithm finds an optimal solution to the "rooted 2-inconnected-subgraph" problem, and moreover, this solution has cost $\leq c(E^*) = \mathrm{opt}$. Hence, $F^{in}(t^*)$ has cost $\leq \mathrm{opt}$. Similarly, $F^{out}(s^*)$ has cost $\leq \mathrm{opt}$. Thus, the total cost of $\widehat{G}$ is $\leq 2\mathrm{opt}$.

We prove that G^* is 2-(S, t^*) connected by a contradiction argument. Suppose that the connectivity requirement does not hold for G^*. Then there is a cut $(U, V - U)$ of size < 2 such that $U \cap S$ is nonempty and $t^* \in V - U$. Since G^* is 2-(S, T) connected, we have $T \subseteq U$. Moreover, G^* has a T, t^* dipath (i.e., a dipath from a vertex of T to t^*) because G^* has the T, S dipath $\widehat{P}$ which contains t^*. Hence, the cut $(U, V - U)$ has one edge of the T, t^* dipath; let this edge be f. Then the digraph $G^* - f$ has no T, t^* dipath. Hence, $G_0 - f$ has no T, S dipath (if such a dipath existed, then it would contain the cut edge e_1 and its tail vertex t^*, thus it would contain a subpath from T to t^*). Thus f is a cut edge, say $f = e_j, j = 1 \ldots, \ell$. This gives a contradiction because $G_0 - e_j$ contains the T, t^* dipath $\widehat{P}(\widehat{t}, t^*)$ for each $j = 1, \ldots, \ell$. $\square$

Suppose that the resulting digraph is not 2-(S, T) connected. Then, there exists an edge whose removal results in a digraph that is not (S, T) connected. The next lemma shows that any such edge must be a cut edge.

Lemma 19. *Consider the current digraph $\widehat{G} = G_0 + F^{in}(t^*) + F^{out}(s^*)$. Suppose that f is an edge such that $\widehat{G} - f$ is not (S, T) connected. Then f is a cut edge.*

Proof. Observe that $\widehat{G} - f$ has a dipath from every vertex of S to t^* (due to $F^{in}(t^*)$), and it has a dipath from s^* to every vertex of T (due to $F^{out}(s^*)$). If there is a t^*, s^* dipath in $\widehat{G} - f$, then $\widehat{G} - f$ would be (S,T) connected and we would get a contradiction. Lemma 17 shows that $G_0 - f$ has a t^*, s^* dipath, unless f is a cut edge. This completes the proof. $\square$

4.5 Last step for $\nu = 1$: "Eliminating" all cut edges

The last part of the algorithm "eliminates" the cut edges; we examine the cut edges $e_1, e_2, \ldots, e_\ell$ and find a set of augmenting edges F'; we will prove that the digraph obtained by adding F' to $\widehat{G}$ is 2-(S,T) connected and it has cost $\leq 3\mathrm{opt}$.

To see the key idea, consider the special case of one cut edge, that is, $\ell = 1$. We apply the algorithm for (S,T) connectivity to $\widehat{G} - e_1$ to find a set of augmenting edges F_1 such that $\widehat{G} - e_1 + F_1$ is (S,T) connected. Observe that $\widehat{G} - e_1$ has no T, S dipaths, hence, in this special case, our algorithm finds a set of augmenting edges F_1 of minimum cost. We claim that $\widehat{G} + F_1$ is 2-(S,T) connected. To prove this, suppose that there exists an edge e whose deletion results in a digraph that is not (S,T) connected. By Lemma 19, e is a cut edge of $\widehat{G}$; thus $e = e_1$. Then we get a contradiction since $\widehat{G} - e_1 + F_1$ is (S,T) connected;

In general, for $\ell \geq 2$, we handle all of the cut edges in one step, by "reducing" the problem (of finding a set of augmenting edges F' such that $\widehat{G} + F' - e_i$ is (S,T) connected, for each $i \in \{1, \ldots, \ell\}$) to a single (S,T) connectivity problem on an auxiliary digraph G'. The auxiliary digraph G' is obtained from $\widehat{G}$ by

(i) deleting all the cut edges $e_1, \ldots, e_\ell$, and

(ii) adding an auxiliary edge (v, w) of zero cost for each pair $v \in S$, $w \in T$ such that $\widehat{G}$ has two edge-disjoint v, w dipaths.

As above, we apply the algorithm for (S,T) connectivity to G' to find a set of augmenting edges F' such that $G' + F'$ is (S,T) connected. Let A' denote the set of auxiliary edges of G' (edges present in G' but not in $\widehat{G}$). The correctness of this method follows from the following lemma.

Lemma 20. *Consider the auxiliary digraph G' and a set of augmenting edges F'. Then*

(i) $G' + F'$ *is* (S,T) *connected implies* $\widehat{G} + F'$ *is* 2-(S,T) *connected, and*

(ii) $G_0 + F'$ *is* 2-(S,T) *connected implies* $G' + F'$ *is* (S,T) *connected.*

Proof. Consider the easy part (i) first. Suppose that $G' + F'$ is (S,T) connected, but $\widehat{G} + F'$ is not 2-(S,T) connected. Then, by Lemma 19, there exists a cut edge $e \in \{e_1, \ldots, e_\ell\}$ such that $\widehat{G} + F' - e$ is not (S,T) connected. Consider any pair $s \in S$, $t \in T$. The augmented auxiliary digraph $G' + F'$ has an s, t dipath P', and P' contains none of the edges in $\{e_1, \ldots, e_\ell\}$, but P' may contain one or more of the auxiliary edges; now, observe that for each auxiliary edge $(v, w) \in A'$, $\widehat{G}$ has at least two edge-disjoint v, w dipaths, hence, $\widehat{G} - e$ has a v, w dipath $P_{v,w}$; thus, the union of $P' - A'$ and $\bigcup_{(v,w) \in A'} P_{v,w}$ contains an s, t dipath of $\widehat{G} - e$. We get the desired contradiction since $\widehat{G} + F' - e$ is (S,T) connected.

Now, consider the other part. Suppose that $G_0 + F'$ is 2-(S,T) connected, but $G' + F'$ is not (S,T) connected.

Then there exists a pair $s \in S$, $t \in T$ such that

(a) $G_0 + F'$ has two edge-disjoint s, t dipaths, but

(b) $G' + F'$ has no s, t dipath, and

(c) $\widehat{G}$ does not have two edge-disjoint s, t dipaths (otherwise, G' would have the auxiliary edge (s, t)).

We derive a contradiction by showing that $\widehat{G}$ has two edge-disjoint s, t dipaths if statements (a) and (b) hold.

Let P_1 and P_2 denote two edge-disjoint s, t dipaths of $G_0 + F'$. One of these dipaths (possibly, both of them) has an augmenting edge, otherwise, both dipaths would be contained in G_0 and thus in $\widehat{G}$. Observe that every dipath of $G_0 + F'$ that avoids all cut edges is contained in $G' + F'$. Since P_1 and P_2 are not contained in $G' + F'$, each of these dipaths must contain a cut edge. Moreover, neither P_1 nor P_2 has two or more augmenting edges; to see this, suppose that P_1 contains an augmenting edge (s_1, t_1) followed by another augmenting edge (s_2, t_2); then, by Lemma 16, all the cut edges $e_1, \ldots, e_\ell$ would occur in $P_1(t_1, s_2)$ between the two augmenting edges; but then P_2 (being edge disjoint from P_1) would not contain any cut edges.

Consider two cases:

(1) one of P_1, P_2, say P_1, contains augmenting edges, but the other one, P_2, contains no augmenting edges;

(2) P_1, P_2 both contain augmenting edges.

The next claim states simple but useful properties of G.

Claim 21. *Consider the digraph G and any dipath P that has an augmenting edge α and a cut edge e. Then either*
(1) the first cut edge following α (in P) is e_1, or
(2) the last cut edge preceding α (in P) is e_ℓ.

Proof. Either α precedes some cut edge, or it follows all cut edges. Suppose the first cut edge following α (in P) is e_i ($i \geq 2$); then the union of $P(\alpha, e_i)$ and $\widehat{P} - e_1$ contains a T, S dipath, a contradiction to the definition of e_1. Similarly, if the last cut edge preceding α (in P) is e_i ($i < \ell$), then we get a contradiction to the definition of e_ℓ. The claim follows. □

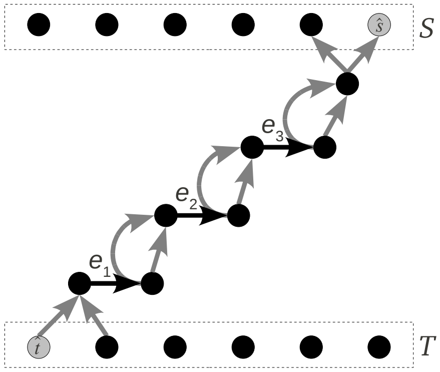
By way of contradiction, assume that $\widehat{G}$ does not have two edge-disjoint s, t dipaths. Then there exists an edge e such that $\widehat{G} - e$ has no s, t dipath.

First, consider case (1): P_1 has augmenting edges, but P_2 has none. If $e \notin P_2$, then we are done because $P_2 = P_2 - e$ is an s, t dipath of $\widehat{G}$. Now, suppose $e \in P_2$. Consider P_1, and let α be its unique augmenting edge. By Claim 21, either $e_1 \in P_1$ or $e_\ell \in P_1$. First, suppose that e_1 is the first cut edge in P_1 following α; thus, the tail t^* of e_1 is in P_1. Observe that $e \notin P_1(t^*, t)$ because P_1, P_2 are edge disjoint and $e \in P_2$. Moreover, the unique augmenting edge of P_1 precedes t^*, hence, $P_1(t^*, t)$ contains no augmenting edges. Finally, observe that $\widehat{G} - e$ has a dipath P'' from s to t^* because $\widehat{G}$ is 2-(S, t^*) connected (by Lemma 18), hence, deleting any edge results in a digraph that is 1-(S, t^*) connected, and thus has an s, t^* dipath. (We state this observation as a claim below, for further use.) It follows that $\widehat{G} - e$ contains the union of P'' and $P_1(t^*, t)$, which contains an s, t dipath.

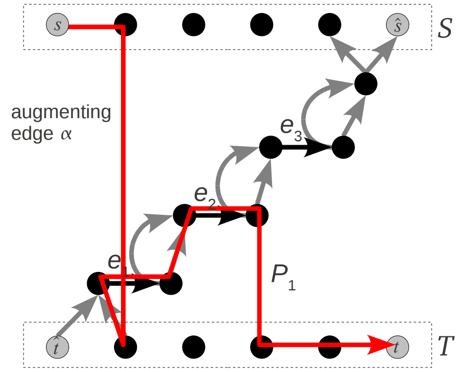
Claim 22. *Let s and t be as above, and let e be any edge of $\widehat{G}$.*
(1) $\widehat{G} - e$ has a dipath P'' from s to t^*, where t^* denotes the tail of e_1.
(2) $\widehat{G} - e$ has a dipath P''' from s^* to t, where s^* denotes the head of e_ℓ.

Now, suppose that e_ℓ is the last cut edge in P_1 preceding α. Then, a similar argument shows that $e \notin P_1(s, s^*)$, and $P_1(s, s^*)$ contains no augmenting edges. Thus, applying Claim 22 (and its notation), $\widehat{G} - e$ contains the union of $P_1(s, s^*)$ and P''', which contains an s, t dipath.

Thus, in Case (1), we get the desired contradiction: $\widehat{G}$ has two edge-disjoint s, t dipaths.



(a) T, S dipath $\widehat{P}$ of G_0.



(b) S, T dipath P_1 of $G_0 + F'$.

Figure 4: The figures illustrate the notation in Lemma 20. The first figure illustrates the T, S dipath $\widehat{P}$ and its cutedges. The second figure illustrates one of the possible S, T dipaths P_1 of $G_0 + F'$.

Finally, consider Case (2): both P_1, P_2 contain augmenting edges and cut edges. Let e be an edge such that $\widehat{G} - e$ has no s, t dipath. By Claim 21, either $e_1 \in P_1$ or $e_\ell \in P_1$, and the same holds for P_2; moreover, neither P_1 nor P_2 contains two or more augmenting edges. We may fix the indices of P_1 and P_2 such that $e_1 \in P_1$ and $e_\ell \in P_2$. Then note that the dipaths $P_1(e_1, t)$ and $P_2(s, e_\ell)$ are edge disjoint, and one of them avoids e. As above, we apply Claim 22 (and its notation). If $P_1(e_1, t)$ avoids e, then the union of P'' and $P_1(e_1, t)$ is contained in $\widehat{G} - e$, and it contains an s, t dipath; otherwise, the union of P''' and $P_2(s, e_\ell)$ is contained in $\widehat{G} - e$, and it contains an s, t dipath. Thus, in Case (2), we get the desired contradiction: $\widehat{G}$ has two edge-disjoint s, t dipaths.

This completes the proof of the lemma. □

The next result summarizes the contributions of this subsection by proving the correctness and the approximation guarantee for the above algorithm.

Lemma 23. *The digraph returned by the algorithm is* 2-(S, T) *connected, and it has cost* $\leq 3\text{opt}$.

Proof. The digraph returned by the algorithm has the edge set $E_0 \bigcup F^{out}(s^*) \bigcup F^{in}(t^*) \bigcup F'$. Each of the sets of augmenting edges has cost $\leq$ opt, hence $\widehat{G}$ has cost ≤ 3opt.

To see the correctness, first note that $\widehat{G}$, which has the edge set $E_0 \bigcup F^{out}(s^*) \bigcup F^{in}(t^*)$, is both 2-(S, t^*) connected and 2-(s^*, T) connected (see Lemma 18). Also, observe that $G_0 + E^*$ is 2-(S, T) connected, hence, by Lemma 20, $G' + E^*$ is (S, T) connected. Thus, the algorithm succeeds in finding a set of augmenting edges F' such that $G' + F'$ is (S, T) connected, and hence (by Lemma 20), $\widehat{G} + F'$ is 2-(S, T) connected. Moreover, $c(F') \leq c(E^*) = \text{opt}$. □

4.6 Combining the cases of 2-(S, T) connectivity

This subsection summarizes our approximation algorithm and analysis for the min-cost 2-(S, T) connectivity problem, see Theorem 3 in Section 1.3.

Proof of Theorem 3. The correctness of the output follows from the correctness proofs of the three main cases in the algorithm.

The cost analysis follows easily from the cost analysis of the three main cases in the algorithm. In the case of no T, S dipaths, the cost of $\widehat{G}$ is $\leq 2\,\mathrm{opt}$. In the case of one T, S dipath, but not two edge-disjoint T, S dipaths, the cost of $\widehat{G}$ is $\leq 3\,\mathrm{opt}$. In the case of two edge-disjoint T, S dipaths, the cost of $\widehat{G}$ is $\leq 3\,\mathrm{opt}$. □

5 An $O(\log k \cdot \log n)$ approximation algorithm for standard k-(S,T) connectivity

5.1 Introduction to k-(S,T) connectivity

This section focuses on the standard version of the k-(S,T) connectivity problem: we are given an integer $k \geq 0$, a directed graph $G = (V, E_0 \cup E)$, two subsets S, T of V, and positive costs on the edges in E; moreover, each edge in E has its tail in S and its head in T. A digraph is called k-(S,T) connected if it has k edge-disjoint dipaths between every vertex $s \in S$ and every vertex $t \in T$. The goal is to find a subset of edges $\widehat{E} \subseteq E$ of minimum cost such that the subgraph $(V, E_0 \cup \widehat{E})$ is k-(S,T) connected.

Although the k-(S,T) connectivity problem pertains to edge-connectivity, it can be seen that the k-VCSS problem is a special case of this problem by applying the reduction in Proposition 7 (but keeping only one copy of each edge of the form (v^-, v^+)). In particular, the k-VCSS digraph has k openly-disjoint dipaths between every pair of vertices v, w iff the digraph resulting from the reduction has k edge-disjoint dipaths between every vertex $v^+ \in S$ and every vertex $w^- \in T$.

The main result of this section is Theorem 4, stated in Section 1.3; it is proved by generalizing the algorithm and analysis for the k-VCSS problem in [9], which in turn is based on ideas and results from [16, 22, 29, 24]. At a high level, our algorithm and analysis are almost the same as the halo-set method of [9]. But there are differences. The application of the halo-set method in [9] relies on the property of "disjointness of cores," whereas our application circumvents this property.

5.2 An approximation algorithm for the k-(S,T) connectivity problem

First, we show that the LP-scaling technique [18] applies in our setting. Based on that, we focus on the key subproblem of increasing (S,T) connectivity by one.

Consider the following LPs: The first one, denoted $\mathrm{LP}(k)$, is a well-known LP relaxation for the k-(S,T) connectivity problem, where E_0 denotes the edge set of the initial digraph, see [12, 15]. The second one, denoted $\mathrm{LP}^{inc}(\ell)$, is a well-known LP relaxation for the problem of increasing (S,T) connectivity by one by adding edges from $E - E_\ell$ to a digraph (V, E_ℓ) that is ℓ-(S,T) connected; we may view (V, E_ℓ) as the "initial digraph."

LP for k-(S,T) connectivity

$$\mathrm{LP}(k) \quad \begin{cases} z_k^* = \min & \displaystyle\sum_{e \in E - E_0} c(e) \cdot \boldsymbol{x}_e \\ \text{s.t.} & \boldsymbol{x}(\delta^{out}_{E-E_0}(U)) + d^{out}_{E_0}(U) \geq k, \quad \forall U \subseteq V,\ U \cap S \neq \emptyset,\ U \cap T \neq T \\ & 0 \leq x_e \leq 1, \quad \forall e \in E - E_0 \end{cases}$$

LP for increasing (S,T) connectivity from ℓ to $\ell + 1$

$$\text{LP}^{inc}(\ell) \quad \begin{cases} z^{inc} = \min & \sum_{e \in E - E_\ell} c(e) \cdot \boldsymbol{x}_e \\ \text{s.t.} & \boldsymbol{x}(\delta^{out}_{E-E_\ell}(U)) \geq 1, \quad \forall U \subseteq V,\ U \cap S \neq \emptyset,\ U \cap T \neq T,\ d^{out}_{E_\ell}(U) = \ell \\ & 0 \leq x_e \leq 1, \qquad\qquad\quad \forall e \in E - E_\ell \end{cases}$$

Proposition 24. *Suppose there is an approximation algorithm for the problem of increasing the (S,T) connectivity of a digraph by one that achieves an approximation guarantee of $\beta(n)$ with respect to the LP relaxation* $\text{LP}^{inc}(\ell)$*. Then there is an $O(\beta(n)\log k)$-approximation algorithm for the k-(S,T) connectivity problem.*

We omit our proof, which follows from the well-known LP-scaling technique; a proof is given in [25].

In the rest of this section, we present our approximation algorithm for increasing the (S,T) connectivity by one. We assume that the initial digraph is ℓ-(S,T) connected. This assumption is valid because previous iterations of the algorithm have increased the (S,T) connectivity from zero to ℓ.

5.3 Preliminaries on ℓ-(S,T) connected digraphs

This subsection develops some basic results on ℓ-(S,T) connected digraphs, where ℓ is a nonnegative integer.

A *deficient set* is a set of vertices $U \subseteq V$ such that $U \cap S \neq \emptyset$, $U \cap T \neq T$, and $d^{out}(U) < \ell + 1$. Thus there exists a pair of vertices $s \in S$ and $t \in T$ such that U "separates" s and t, so any feasible solution of the $(\ell+1)$-(S,T) connectivity problem has $\geq \ell + 1$ edges in the cut $(U, V-U)$, but the current digraph has $\leq \ell$ edges in the cut. Observe that every deficient set U has $d^{out}(U) = \ell$, since we assume that the initial digraph is ℓ-(S,T) connected. The next lemma is basic and it follows from submodularity and the assumption on the initial digraph; see [25] for a proof.

Lemma 25 (Uncrossing Lemma)**.** *Let U and W be two deficient sets such that $(U \cap W) \cap S \neq \emptyset$ and $(U \cup W) \cap T \neq T$. Then both $U \cap W$ and $U \cup W$ are deficient sets.*

We call an inclusionwise minimal deficient set a *core*, and denote it by C, or C_i, etc. The *halo family* of a core C, denoted $\text{Halo}(C)$, is the family of deficient sets containing C but containing no other cores, that is,

$$\text{Halo}(C) = \{U : U \text{ is a deficient set }, C \subseteq U, U \text{ contains no other cores}\}.$$

The *halo set* of C, denoted $\text{H}(C)$, is the union of all members of the halo family of C, that is, $\text{H}(C) = \bigcup\{W : W \in \text{Halo}(C)\}$.

We say that an edge $e = (v,w)$ *covers* a deficient set U if e has its tail in U and its head in $V - U$. Similarly, we say that a set of edges F *covers* $\text{Halo}(C)$ if every member of $\text{Halo}(C)$ is covered by some edge in F.

For a deficient set U, we define the *body* to be $U \cap S$, and we define the *shadow* to be $T - U$. The next lemma is a key tool for our algorithm and its analysis.

Lemma 26 (Disjointness Property)**.** *Let C and D be two distinct cores. Let U be a deficient set in* $\text{Halo}(C)$*, and let W be a deficient set in* $\text{Halo}(D)$*. Then either*

- *$(S \cap U)$ and $(S \cap W)$ are disjoint, or*
- *$(T - U)$ and $(T - W)$ are disjoint.*

Proof. Suppose that $(S \cap U)$ and $(S \cap W)$ intersect; otherwise, the lemma holds. For the sake of contradiction, suppose that $(T - U)$ and $(T - W)$ intersect. Then we have

$$(S \cap U) \cap (S \cap W) = (U \cap W) \cap S \neq \emptyset$$
$$(T - U) \cap (T - W) = T - (U \cup W) \neq \emptyset.$$

Then by Lemma 25, $U \cap W$ is a deficient set, and thus it contains a core. We have a contradiction because C is the unique core contained in U, D is the unique core contained in W, and C, D are distinct. $\square$

5.4 Computing cores

This subsection describes an efficient algorithm for computing all of the cores. Recall our assumption that the current digraph is ℓ-(S, T) connected.

For each pair of vertices $s \in S, t \in T$, we apply an efficient max s, t-flow min s, t-cut algorithm to find a smallest set of vertices $C_{s,t}$ that induces a minimum s, t-cut. It can be seen that if the value of the maximum flow is less than $\ell + 1$ (the required (S, T) connectivity), then $C_{s,t}$ is the unique minimal deficient set that includes s and excludes t, thus $C_{s,t}$ is a candidate core; otherwise, there exists no deficient set (and no core) that includes s and excludes t; clearly, $C_{s,t}$ is not a core if it properly contains another set $C_{s',t'}$, where $s' \in S, t' \in T$, but otherwise, $C_{s,t}$ is a core. Finally, we construct a family $\mathcal{C}$ by choosing every subset $C_{s,t}$ that does not properly contain another set $C_{s',t'}$, where $s' \in S, t' \in T$. The family $\mathcal{C}$ is the family of all the cores. Moreover, the construction immediately implies an upper bound of $|S| \cdot |T|$ on the number of cores, and there exist examples showing that this bound is tight. A proof of the next result is given in [25].

Proposition 27. *For every pair of vertices $s \in S, t \in T$, if the above algorithm finds a set $C_{s,t}$ then the set is the unique minimal deficient set that includes s and excludes t. Moreover, the algorithm finds all of the cores by computing $\mathcal{C}$. The number of cores is at most $|S| \cdot |T|$.*

5.5 Covering a halo family via Frank's algorithm

A key subroutine of our algorithm uses an algorithm due to Frank [13] to cover the halo family of a core.

Consider a core C, and the halo family of C. To cover the halo family, we first add so-called padding edges that cover all deficient sets that are not in the chosen halo family. In particular, for *each* core $D \neq C$, we choose an arbitrary vertex $u_D \in D \cap S$ and add new edges from u_D to each vertex $v \in (T - D)$; thus, the set of new edges for the core D is $\{(u_D, w) : w \in (T - D)\}$; we call these edges the *padding edges*. After adding all the padding edges, we choose an arbitrary root vertex $r_C \in C \cap S$ and run Frank's algorithm on the resulting digraph, with r_C as the root vertex and T as the set of terminals; the set of augmenting edges E stays the same, and the initial digraph has all the edges of the original initial digraph G_0 as well as all of the padding edges. We claim that the set of augmenting edges $F(C)$ computed by this algorithm covers the halo family of our chosen core C. A proof of the next result is given in [25].

Proposition 28. *Let C be the chosen core. Then the set of augmenting edges $F(C)$ found by Frank's algorithm covers the halo family of C.*

Here, we discuss another way of covering a halo family via Frank's algorithm. A family of sets $\mathcal{F}$ is called a *T-intersecting family* if, for any pair of sets $U, W \in \mathcal{F}$, if $U \cap W \cap T \neq \emptyset$, then both $U \cap W$ and $U \cup W$ are also in $\mathcal{F}$. Given a set of augmenting edges that all have heads in a set T, Frank's algorithm finds a minimum-cost set of augmenting edges that covers a T-intersecting family. For a core C, the family

$\mathcal{F} = \{V - U : U \in \text{Halo}(C)\}$ forms a T-intersecting family; this follows from Lemma 29. Hence, Frank's algorithm can be applied to find a subset $F(C)$ of augmenting edges of minimum cost that covers $\mathcal{F}$, and thus covers $\text{Halo}(C)$.

Lemma 29. *Let C be a core. Then the family $\mathcal{F} = \{V - U : U \in \text{Halo}(C)\}$ forms a T-intersecting family.*

Proof. Consider a pair of deficient sets $U, W \in \text{Halo}(C)$. Suppose that the complements intersect in T, that is, $V - U$ and $V - W$ intersect in T. Then, $(U \cup W) \cap T \neq T$. Clearly, every pair of deficient sets $U, W \in \text{Halo}(C)$ intersect in S. Thus, by Lemma 25 (uncrossing), both $U \cup W$ and $U \cap W$ are deficient sets. Moreover, it can be seen that $U \cup W$ and $U \cap W$ are in $\text{Halo}(C)$, if both sets are deficient sets; therefore, $V - (U \cup W)$ and $V - (U \cap W)$ are in $\mathcal{F}$. Hence, if sets $V - U$ and $V - W$ from our family $\mathcal{F}$ intersect in T, then both $(V - U) \cap (V - W)$ and $(V - U) \cup (V - W)$ are in $\mathcal{F}$. $\square$

In the proof of Lemma 32 (see below), we need the property that Frank's algorithm does not cover any other core when it is used to cover the halo family of a core C; we prove this in the next result.

Lemma 30. *Let C be a core, and let $F(C)$ be an (inclusionwise) minimal set of augmenting edges that covers* $\text{Halo}(C)$*. Let $D \neq C$ be another core. Then no edge in $F(C)$ covers D.*

Proof. By way of contradiction, suppose that D is covered by some edge $e \in F(C)$, where $e = (v, q)$. By the minimality of $F(C)$, e covers at least one deficient set $U \in \text{Halo}(C)$. Since the edge $e = (v, q)$ covers both U and D, its tail v is in $(S \cap U) \cap (S \cap D)$ and its head q is in $(T - U) \cap (T - D)$. This is a contradiction by Lemma 25 (uncrossing) since $U \cap D$ is a deficient set that is properly contained in D. $\square$

5.6 Approximation algorithms for increasing (S, T) connectivity

We increase the (S, T) connectivity by iteratively adding edges of low cost to decrease the number of cores until no cores are left; if there are no cores, then observe that the (S, T) connectivity of the digraph has increased by one.

We present two different algorithms that yield the same approximation guarantee up to constant factors. The first algorithm follows a sequential greedy strategy, and it achieves an approximation guarantee of $H_{|S| \cdot |T|} = O(\ln |S| \cdot |T|)$, where H_ℓ denotes the ℓth harmonic number. The second algorithm has a better running time, and it achieves an approximation guarantee of $O(\log_2 |S|)$. The sequential greedy strategy of the first algorithm has been used earlier for the k-VCSS problem by [9], and the parallel strategy of the second algorithm has been used earlier for the k-VCSS problem by [26]. Both algorithms rely on Frank's algorithm; in general, the set of augmenting edges computed by Frank's algorithm is *not* added to the current digraph; instead, we compute the cost of this edge set, and if it satisfies other criteria, then we add this edge set to the current digraph.

Proposition 31. *There is an $O(\log n)$-approximation algorithm for increasing (S, T) connectivity from ℓ to $\ell + 1$.*

The main result of this section, Theorem 4, follows from the above result and Proposition 24.

5.7 Approximation Algorithm 1

Our first algorithm decreases the number of cores by one in each iteration. Consider any iteration: For each core C, we apply Frank's algorithm to compute a set of edges $F(C)$ that covers the halo family of C; but, at this point, we do not add any edges to the current digraph. We then choose a core C^* such that $c(F(C^*))$ is minimum, that is, $c(F(C^*)) = \min\{c(F(C)) : C \text{ is a core}\}$, and we add $F(C^*)$ to the current digraph. Lemma 32 below shows that the number of cores decreases by one in the resulting digraph. We repeat these iterations until no core is left in the current digraph.

In general, when we add some augmenting edges, we cover some of the old cores, but the augmented digraph may have several new cores that are intersecting, e.g., there may exist $j \geq 2$ new cores that intersect each other but whose union contains less that j old cores; see Figure 5. (Such complications do not arise in the algorithm of [9] for the k-VCSS problem since the cores are disjoint in [9].)

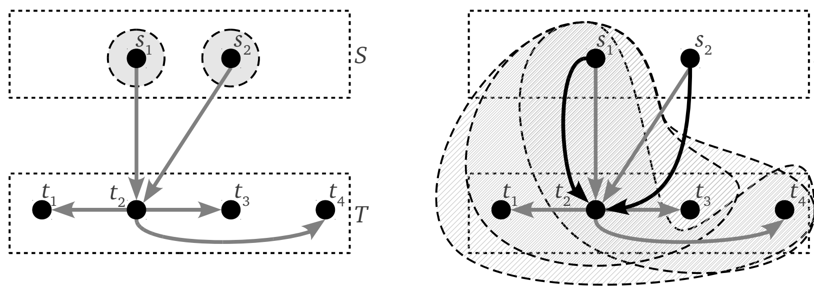


Figure 5: The figure shows an example where the number of cores increases after adding augmenting edges; these edges are indicated by black lines. The problem is to increase (S, T)-connectivity by one, where the initial digraph is 1-(S, T)-connected; the edges of the initial digraph are indicated by grey lines. The example in the left figure has two cores, $\{s_1\}$ and $\{s_2\}$. The example in the right figure is obtained by adding the augmenting edges (s_1, t_2) and (s_2, t_2); it has six cores: $\{s_1, t_1, t_2, t_3\}$, $\{s_1, t_1, t_2, t_4\}$, $\{s_1, t_2, t_3, t_4\}$, $\{s_2, t_1, t_2, t_3\}$, $\{s_2, t_1, t_2, t_4\}$, and $\{s_2, t_2, t_3, t_4\}$.

Lemma 32. *If we cover the halo family of a core C (by adding the edge set $F(C)$ computed by Frank's algorithm), then the number of cores decreases by at least one.*

Proof. We refer to the cores in the "old digraph" $(V, E_0 \cup E')$ as the *old cores*, and the cores in the "new digraph" $(V, E_0 \cup E' \cup F(C))$ as the *new cores*.

It can be seen that the lemma follows from two key facts: (1) every one of the deficient sets in $\text{Halo}(C)$ is covered by the set of augmenting edges $F(C)$; (2) every one of the old cores other than C is preserved, that is, except for C, all of the old cores are new cores. Fact (1) holds by definition; we will prove fact (2) below.

Consider fact (2) and its proof. When Frank's algorithm is applied to any core C, then it finds an (inclusionwise) minimal set of augmenting edges $F(C)$ that covers $\text{Halo}(C)$; the minimality holds because the algorithm finds a set of augmenting edges of minimum cost. It then follows from Lemma 30 that for any core $D \neq C$, D is not covered by any edge in $F(C)$. Thus every old core $D \neq C$ stays as a deficient set of the new digraph, and moreover, it must be a new core. $\square$

5.8 Cost analysis by decomposing an optimal fractional solution

For the rest of this section, we revise our definitions of opt and E^*, for the sake of notational convenience. We use opt to denote the optimal value of $\mathrm{LP}^{inc}(\ell)$, which is the LP relaxation for increasing (S, T) connectivity from ℓ to $\ell+1$, and we use E^* to denote the support of some fixed optimal solution of this LP (thus, letting $\boldsymbol{x}$ denote an optimal solution of $\mathrm{LP}^{inc}(\ell)$, we have $E^* = \{e \in E : \boldsymbol{x}_e > 0\}$).

Let $C_1, C_2, \ldots, C_t$ denote all of the cores. For each core C_i, $1 \le i \le t$, let $E^*(C_i)$ denote an (inclusion-wise) minimal subset of E^* such that $\mathrm{Halo}(C_i)$ is covered by $\boldsymbol{x}$ restricted to $E^*(C_i)$.

Lemma 33 (Decomposition Lemma)**.** *$E^*(C_i)$ and $E^*(C_j)$ are disjoint for all $1 \le i \ne j \le t$. Furthermore, $\sum_{i=1}^{t} \sum_{e \in E^*(C_i)} c(e)\boldsymbol{x}(e) \le \mathrm{opt}$.*

Proof. We prove the first statement by a contradiction argument. Suppose that the statement does not hold. Then there exist i, j with $1 \le i < j \le t$, such that $E^*(C_i) \cap E^*(C_j)$ contains an augmenting edge e. Then by the minimality of $E^*(C_i)$ and $E^*(C_j)$, e must cover some deficient set $U \in \mathrm{Halo}(C_i)$ as well as some deficient set $W \in \mathrm{Halo}(C_j)$. This contradicts Lemma 26 (the disjointness property). Hence, $E^*(C_i)$ and $E^*(C_j)$ are disjoint for all $i \ne j$.

The second statement is an immediate consequence of the first statement. □

Lemma 34. *Let t be the number of cores. Let $C_1, \ldots, C_t$ be all of the cores, and let $F(C_i)$ be an edge set of minimum cost that covers $\mathrm{Halo}(C_i)$, $\forall i = 1, \ldots, t$. Then $\sum_{i=1}^{t} c(F(C_i)) \le \mathrm{opt}$.*

Proof. Consider any core C. Recall that $F(C)$ denotes the set of augmenting edges found by Frank's algorithm, and $E^*(C)$ denotes an (inclusion-wise) minimal subset of E^* such that $\mathrm{Halo}(C)$ is covered by $\boldsymbol{x}$ restricted to $E^*(C)$. Then we have $c(F(C)) \le \sum_{e \in E^*(C)} c(e)\boldsymbol{x}(e)$. This follows from Frank's results on the LP relaxation for the min-cost k-(r, T) connected digraph problem. Frank proves that the LP relaxation is integral, see Theorem 9, and also see Theorems 4.4 and 5.9 of [14].

We apply this to all the cores $C_1, C_2, \ldots, C_t$. Thus, we have $c(F(C_i)) \le \sum_{e \in E^*(C_i)} c(e)\boldsymbol{x}(e)$, for $i = 1, 2, \ldots, t$. Moreover, we have $\sum_{i=1}^{t} \sum_{e \in E^*(C_i)} c(e)\boldsymbol{x}(e) \le \mathrm{opt}$, by Lemma 33. Hence, $\sum_{i=1}^{t} c(F(C_i)) \le \sum_{i=1}^{t} \sum_{e \in E^*(C_i)} c(e)\boldsymbol{x}(e) \le \mathrm{opt}$. □

Lemma 35. *The total cost incurred by the algorithm is $O(\log n)\mathrm{opt}$.*

Proof. Let t_0 denote the number of cores at the start of the algorithm; we have $t_0 \le |S| \cdot |T| \le n^2$, by Proposition 27. Each iteration decreases the number of cores by one, by Lemma 32. Moreover, we claim that the cost of the set of augmenting edges added by an iteration is $\le \mathrm{opt}/t$, where t denotes the number of cores at the start of the iteration. To see this, consider a core C^* such that the edge set $F(C^*)$ has minimum cost, i.e., $c(F(C^*)) \le c(F(C)), \forall$ cores C. Then $c(F(C^*)) = \min_{i=1}^{t}\{c(F(C_i))\} \le \frac{1}{t}\sum_{i=1}^{t}\{c(F(C_i))\} \le \frac{1}{t}\mathrm{opt}$, where the last inequality follows from Lemma 34.

Therefore, the total cost of the edges added by the algorithm is $\le \mathrm{opt}\ \left(\frac{1}{t_0} + \frac{1}{t_0 - 1} + \ldots + 1\right) = \mathrm{opt}\ H_{t_0} = O(\ln t_0)\ \mathrm{opt} = O(\log n)\ \mathrm{opt}$, where H_ℓ denotes the ℓth harmonic number. □

Approximation Algorithm 1 together with its analysis gives a proof of Proposition 31.

Remark 36. Approximation Algorithm 1 can be implemented to run in time $O(n^6 m + n^5 m \cdot f(m, n))$, where $f(m, n)$ denotes the time for computing a maximum s, t flow.

5.9 Approximation Algorithm 2

The second approximation algorithm executes $O(\log n)$ rounds, where each round adds a set of augmenting edges with the hypothetical goal of decreasing the number of cores by a factor of two. At the start of each round, we compute the set $\mathcal{C}$ of all cores for the current digraph; then, for each core $C \in \mathcal{C}$, we compute the set of edges $F(C)$ that covers $\mathrm{Halo}(C)$, via Frank's algorithm; then, we add all these edge sets to the current digraph, that is, we add the edge set $\bigcup\{F(C) \mid C \in \mathcal{C}\}$; this completes one round. We repeatedly apply such rounds until there is no core left.

Lemma 37. *No deficient set contains two cores whose bodies are intersecting.*

Proof. By Lemma 26 (the disjointness property), any two distinct cores C and D whose bodies are intersecting must have disjoint shadows. Hence, $C \supseteq T - D$, and $D \supseteq T - C$, thus $C \cup D$ contains T. Thus, any set of vertices containing $C \cup D$ cannot be a deficient set, because it contain T. $\square$

Lemma 38. *In each iteration, the maximum number of body-disjoint cores decreases by a factor of two.*

Proof. Let ν and ν' denote the maximum number of body-disjoint cores at the beginning and at the end of the iteration, respectively. We refer to cores at the beginning of the iteration as old cores, and those at the end of the iteration as new cores. In each iteration, the algorithm covers every deficient set that is contained in some halo family. Thus, the current digraph has no deficient set that contains exactly one old core. In other words, any new core contains at least two old cores. By Lemma 37, a new core cannot contain two old cores whose bodies are intersecting because each new core is a deficient set in the old digraph. Hence, ν' body-disjoint new cores must contain at least $2\nu'$ body-disjoint old cores. Thus, $2\nu' \leq \nu$ which proves the lemma. $\square$

Lemma 39. *The algorithm terminates within $O(\log n)$ rounds, and it runs in polynomial time. Moreover, the total cost incurred by the algorithm is at most $O(\log n)\mathrm{opt}$.*

Proof. The maximum number of body-disjoint cores is $O(|S|) = O(n)$, and the maximum number of body-disjoint cores decreases by half in each round, hence the number of rounds is $O(\log n)$.

The cost of the edges added in each round is at most opt. To see this, let $C_1, C_2, \ldots, C_t$ be all of the cores. Recall that, for $i = 1, 2, \ldots, t$, $F(C_i)$ is a set of edges of minimum cost that covers $\mathrm{Halo}(C_i)$, hence, by Lemma 34, we have $\sum_{i=1}^{t} c(F(C_i)) \leq \mathrm{opt}$. Thus the total cost incurred in $O(\log n)$ rounds is $O(\log n)\mathrm{opt}$. $\square$

Approximation Algorithm 2 together with its analysis gives another proof of Proposition 31.

Remark 40. Approximation Algorithm 2 can be implemented to run in time $O((n^4 m + n^3 m \cdot f(m,n)) \cdot \log n)$, where $f(m,n)$ denotes the time for computing a maximum s,t flow.

6 An approximation algorithm for relaxed 1-(S,T) connectivity

In this section, we present an approximation algorithm for the relaxed (S,T) connectivity problem, and prove part (1) of Theorem 5 in Section 1.3. The problem reduces to the special case where there is no T,S dipath. Hence, we focus on this special case. Our approximation algorithm and its analysis are based on a key structural result that decomposes any feasible solution into a set of junction trees that are disjoint on the vertices of T. Our algorithm achieves an approximation guarantee of $\alpha(n)+1$, where $\alpha(n)$ denotes the best available approximation guarantee for the directed Steiner tree problem. Our approximation guarantee is tight up to an additive term of one, since Proposition 1 shows that the relaxed (S, T) connectivity problem is at least as hard as the directed Steiner tree problem.

There is a simple, linear-time reduction from the relaxed (S, T) connectivity problem to its special case where there is no T, S dipath. For each vertex $s \in S$, we add a new vertex s^+ and a new edge (s^+, s) to G_0 (the initial digraph); the vertex s and its other incident edges stay the same. Then we replace each vertex s in S by the associated vertex s^+, to get $S^{new} = \{s^+ : s \in S\}$. It is easily seen that a set of edges $\widehat{E} \subseteq E$ is a solution to the new instance iff it is a solution to the original instance. Observe that the new instance has no T, S^{new} dipath, because each of the vertices s^+ in S^{new} has indegree zero.

6.1 Relaxed (S, T) connectivity: An approximation algorithm for the case of no T, S dipath

In this section, we assume that there is no T, S dipath in G. We start with a key structural result on decomposing a feasible solution. We need the notion of junction trees. Let r be a vertex. Recall that an in-tree J^{in} rooted at r is an edge-minimal digraph that has a v, r dipath for every vertex $v \in V(J^{in})$, and similarly, we have the notion of an out-tree J^{out} rooted at r. A *junction tree J rooted at r* is the union of an in-tree J^{in} and an out-tree J^{out}, both rooted at the same vertex r; the in-tree and the out-tree may have common edges; see [7, 4].

6.1.1 Decomposing a feasible solution

We give a structural result that applies to any feasible solution of the relaxed (S, T) connectivity problem. We prove our approximation guarantee by applying this result to an optimal solution, hence, we consider an optimal solution $G^* = (V, E_0 \cup E^*)$ to the relaxed (S, T) connectivity problem. But, we remark that the results in this subsection (Lemmas 41 and 42) apply for any feasible solution.

We construct junction trees $J_1, J_2, \ldots, J_\ell \subseteq G^*$ with the following properties (see Lemma 42 and its proof, given below):

- For $i = 1, 2, \ldots, \ell$, J_i contains S, and J_i has an s, t dipath for all $s \in S$ and all $t \in V(J_i) \cap T$.
- For $i \neq j$, J_i and J_j have no common vertices of T and thus no common augmenting edges.
- $\bigcup_{i=1}^{\ell} J_i$ contains T; in particular, $\bigcup_{i=1}^{\ell} J_i$ is (S, T) connected.

Intuitively, given an optimal set of augmenting edges E^*, we want to partition it into subsets $E_1^*, \ldots, E_\ell^*$ such that each subset E_i^* together with E_0 forms a junction tree J_i connecting S and $V(J_i) \cap T$.

We start our construction by contracting all maximal strongly-connected components of G^*. Observe that no vertices of S and T are in the same strongly-connected component because G has no T, S dipath; moreover, any two maximal strongly-connected components have no common vertices; otherwise, the two would have been merged. We abuse the notation and continue using the same symbols for the contracted digraph. At this point, the contracted digraph G^* is acyclic. Hence, there exists a vertex $t^* \in T$ such that there exists no t, t^* dipath for any other vertex $t \in T$. We call such a vertex t^* a *top-vertex*.

The construction runs in several iterations on the contracted digraph. In each iteration i, we construct a junction tree J_i whose in-tree contains S and the out-tree contains some vertices of T. We take the root of the junction tree to be a top-vertex t_i of the current digraph. The out-tree of J_i consists of dipaths from t_i to all vertices of T reachable from t_i in the current digraph. Then, we remove the vertices of T that are assigned to J_i from the current digraph. We repeat this process until each vertex of T is assigned to some junction tree.

In more detail, we start with the contracted digraph $G_0^* = G^*$ and the terminal set $T_0 = T$. At the iteration i, for $i = 1, 2, \ldots, \ell$, we consider the digraph G_{i-1}^* and the terminal set T_{i-1}. We choose a top-vertex t_i as the root of the junction tree J_i, and J_i consists of an in-tree J_i^{in} and an out-tree J_i^{out}, that is, $J_i = J_i^{in} \cup J_i^{out}$. Both J_i^{in} and J_i^{out} are subgraphs of G_{i-1}^*. The in-tree J_i^{in} is obtained by taking an in-directed Steiner tree of G_{i-1}^* rooted at t_i with terminal set S. The out-tree J_i^{out} is obtained by taking the union of t_i, t dipaths for all vertices $t \in T_{i-1}$ reachable from t_i in G_{i-1}^*. Once we have the junction tree J_i, we update the digraph G_{i-1}^* and the terminal set T_{i-1} by removing all vertices of T assigned to J_i. Thus, we have $G_i^* = G_{i-1}^* - V(J_i) \cap T$ and $T_i = T_{i-1} - V(J_i) \cap T$. We continue to the next iteration and repeat the process until all vertices of T are assigned to junction trees. The stopping condition is $T_\ell = \emptyset$, or equivalently, T is contained in $\bigcup_{i=1}^{\ell} J_i$.

At termination, we uncontract the strongly-connected components of G^*. Suppose that the root t_i of a junction tree J_i in the contracted digraph corresponds to a nontrivial strongly-connected component C_i; then, when we uncontract the digraph, we take the root t_i to be any vertex of T in C_i. It can be seen that this preserves the three properties of junction trees listed above. We remark that the in-tree and the out-tree may have common vertices of T in the uncontracted digraph, hence, we may have $c(J_i) < c(J_i^{in}) + c(J_i^{out})$.

Clearly, the junction trees $J_1, J_2, \ldots, J_\ell$ of the uncontracted digraph have no common vertices of T, and so they have no common augmenting edges. This implies that $\sum_{i=1}^{\ell} c(J_i) = c(E^*) = \text{opt}$.

The next two results prove that these junction trees satisfy the required properties.

Lemma 41. *At the iteration i, $i = 1, 2, \ldots, \ell$, the digraph G_{i-1}^* is (S, T_{i-1}) connected.*

Proof. The proof hinges on a key property of a junction tree J_i: every node of $V(J_i) \cap T$ is reachable from the root t_i.

We proceed by induction on i for $i = 1, 2, \ldots, \ell$.

Base case $i = 1$: The base case is trivial because the starting digraph $G_0^* = G^*$ is obtained from a feasible solution.

Inductive step $i > 1$: Assume that the induction hypothesis holds for some $i \geq 1$. We will prove that the digraph G_i^* is (S, T_i) connected. Suppose not. Then there exists a pair of vertices $s \in S$ and $t \in T_i$ such that G_i^* has no s, t dipath. By the induction hypothesis, G_{i-1}^* is (S, T_{i-1}) connected and so has an s, t dipath P. Then P must contain some vertex of $V(J_i) \cap T$, otherwise, P is also contained in $G_i^* = G_{i-1}^* - V(J_i) \cap T$. Then G_{i-1}^* has a dipath from t_i to t, because J_i has dipaths from t_i to each vertex of $V(J_i) \cap T$, and so the union of J_i and P contains a t_i, t dipath. By the construction of J_i, the vertex t must be included in J_i and thus must have been removed from G_i^*, a contradiction. Therefore, G_i^* is (S, T_i) connected. $\square$

Lemma 42. *The following properties holds for $J_1, J_2, \ldots, J_\ell$.*

(i) For $i = 1, 2, \ldots, \ell$, J_i contains S, and J_i has an s, t dipath for all $s \in S$ and all $t \in V(J_i) \cap T$. Moreover, if the original digraph G is acyclic on T, then J_i^{in} has exactly one vertex of T, namely, the root t_i.

(ii) For $i \neq j$, J_i and J_j have no common vertices of T and thus no common augmenting edges.

(iii) $\bigcup_{i=1}^{\ell} J_i$ contains T; in particular, $\bigcup_{i=1}^{\ell} J_i$ is (S, T) connected.

Proof. (i) The first property follows from Lemma 41. Consider any $i = 1, 2, \ldots, \ell$. Since G^*_{i-1} is (S, T_{i-1}) connected, it must contain an in-directed Steiner tree J_i^{in} rooted at t_i with terminal set S. Moreover, J_i^{out} has a t_i, t dipath, for every vertex $t \in V(J_i^{out}) \cap T$.

Now, focus on the contracted digraph obtained by contracting all maximal strongly-connected components of G^*. We claim that t_i is the unique vertex of T in J_i^{in}, that is, $V(J_i^{in}) \cap T = \{t_i\}$. Note that J_i^{in} is an in-directed Steiner tree in G^*_{i-1} rooted at t_i. Hence, if J_i^{in} contains some other vertex $t' \in T$, then it has a t', t_i dipath and so does G^*_{i-1}. This is a contradiction since t_i is a top-vertex of G^*_{i-1}, that is, G^*_{i-1} has no dipath from $(T_{i-1} - \{t_i\})$ to t_i.

In general, the original (uncontracted) digraph G^* may have two or more vertices of T in J_i^{in}. In the special case where the digraph G is acyclic on T, observe that every strongly-connected component has at most one vertex of T. Hence, in this special case, the above property of the contracted digraph is preserved even after we uncontract the strongly-connected components, that is, t_i is the unique vertex of T in J_i^{in}.
(ii) The second property holds because we remove all vertices of $V(J_i) \cap T$ from the digraph G^*_{i-1} before proceeding to the next iteration, for $i = 1, 2, \ldots, \ell$. Moreover, no two junction trees have a common augmenting edge, because each vertex of T is in exactly one junction tree, and all augmenting edges have heads in T.
(iii) The last property holds because we stop the construction when $T_\ell = \emptyset$. Hence, by property (i), we have that $\bigcup_{i=1}^{\ell} J_i$ is (S, T) connected. □

6.1.2 An approximation algorithm

Our algorithm constructs an auxiliary digraph, then computes a rooted out-branching M of minimum-cost in it, and then maps M back to the original digraph G to get a solution to the relaxed (S, T) connectivity problem. The auxiliary digraph is constructed as follows. For each vertex $t \in T$, we compute an in-directed Steiner tree F_t rooted at t with terminal set S of approximately minimum cost. Then we remove all incident edges from S, contract S to a single vertex $\widehat{s}$, and add an edge $(\widehat{s}, t)$ of cost $c(F_t)$ for each $t \in T$; finally, we remove all optional vertices $v \in V - (S \cup T)$ that are not reachable from T; this completes the construction of the auxiliary digraph. Then we compute a min-cost out-branching M with root $\widehat{s}$ for the auxiliary digraph; observe that all vertices are reachable from $\widehat{s}$, by construction. Finally, we replace each edge $(\widehat{s}, t)$ of M by the corresponding directed Steiner tree F_t to get a solution digraph $\widehat{G}$. We analyze the cost of $\widehat{G}$ by comparing it to the cost of an optimal solution $G^* = G_0 \cup E^*$. Figure 6 illustrates the working of our algorithm.

The next result gives the correctness and cost analysis for the algorithm.

Proposition 43. *The above algorithm finds a feasible solution of cost* $\leq (\alpha(n) + 1)\mathrm{opt}$ *for the relaxed* (S, T) *connectivity problem.*

Proof. The correctness of our solution follows from the fact that M is an out-branching rooted at $\widehat{s}$. In more detail, consider any vertex $t \in T$. Observe that every $\widehat{s}, t$ dipath in M is of the form $\widehat{s} \to t^* \to \ldots \to t$, where $(\widehat{s}, t^*)$ is an auxiliary edge while the other edges belong to the digraph G. Since we replace $(\widehat{s}, t^*)$ by the in-directed Steiner tree F_{t^*} in the final step, the resulting digraph $\widehat{G}$ must have an s, t^* dipath for every $s \in S$. Hence, we have an s, t dipath of the form $s \to \ldots \to t^* \to \ldots \to t$, for each vertex $s \in S$. Thus, the resulting digraph $\widehat{G}$ is (S, T) connected.

Now, consider the cost analysis. We abbreviate $\alpha(n)$ to α within this proof. We have $c(\widehat{E}) \leq c(M)$. Our key claim is that $c(M) \leq (\alpha + 1)\mathrm{opt}$. To prove this, we start with the digraph $G^* = (V, E_0 \cup E^*)$ of an optimal solution E^* and construct a spanning subgraph M^* of G^{aux} such that M^* contains an out-branching

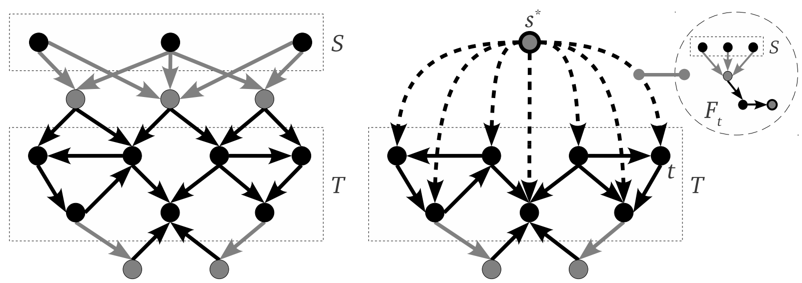


Figure 6: The figure illustrates the working of our approximation algorithm for the relaxed (S, T) connectivity problem on an example that has no T, S dipath. The left figure shows the original digraph. The right figure shows the auxiliary digraph. The set of vertices S is contracted into a single vertex $\widehat{s}$. The black vertices are vertices of S and T. The grey vertices are optional vertices. The grey lines denote edges of the initial digraph. The black lines denote augmenting edges. The dash-lines denote auxiliary edges obtained by replacing an in-directed Steiner tree F_t rooted at a vertex $t \in T$ by an edge $(\widehat{s}, t)$ with cost $c(F_t)$ for each vertex $t \in T$.

of G^{aux} rooted at $\widehat{s}$ and $c(M^*) \leq (\alpha + 1)\mathrm{opt}$; clearly, this will prove the claim. We apply Lemma 42 to G^* to obtain junction trees $J_1, \ldots, J_\ell$ that satisfy properties (i)–(iii) of the lemma. Recall that t_i denotes the root of the in-directed Steiner tree J_i^{in} of J_i, for $i = 1, \ldots, \ell$. For each of the junction trees J_i, where $i = 1, \ldots, \ell$, we add the auxiliary edge $(\widehat{s}, t_i)$ to M^*; observe that $c(\widehat{s}, t_i) \leq \alpha\, c(J_i^{in})$ because J_i^{in} is an in-directed Steiner tree of G rooted at t_i with terminal set S and $c(\widehat{s}, t_i) \leq \alpha\, c^*_{t_i}$, where $c^*_{t_i}$ denotes the minimum cost of an in-directed Steiner tree of G rooted at t_i with the same terminal set as J_i^{in}.

Next, for $i = 1, \ldots, \ell$, we add to M^* all the edges of J_i^{out}. Observe that J_i^{out} is a subgraph of G^{aux} because each edge in J_i^{out} is reachable from t_i, hence, both end-vertices of the edge are in $T \cup Q$. Since the junction trees $J_1, \ldots, J_\ell$ satisfy properties (i)–(iii) (of Lemma 42), it follows that M^* contains T, and moreover, M^* has an $\widehat{s}, t$ dipath for each vertex $t \in T$.

Finally, we add to M^* the edges of a minimum-cost dipath from T to v, for each optional vertex $v \in Q$; each of these edges has cost zero since the head is in Q. This completes the construction of M^*. Clearly, M^* contains an out-branching of G^{aux} rooted at $\widehat{s}$. We have

$$c(M^*) \leq \sum_{i=1}^{\ell} (\alpha\, c(J_i^{in}) + c(J_i^{out})) \leq \sum_{i=1}^{\ell} (\alpha\, c(J_i) + c(J_i)) \leq (\alpha + 1) \sum_{i=1}^{\ell} c(J_i) \leq (\alpha + 1)\mathrm{opt}.$$

This implies that our algorithm achieves an approximation guarantee of $(\alpha(n) + 1)$ as required. We remark that the additive term of $+1$ arises because $c(J_i)$ may be strictly less than $c(J_i^{in}) + c(J_i^{out})$, since J_i^{in} and J_i^{out} may have common vertices of T. $\square$

6.2 The relaxed (S, T) connectivity problem on a digraph that is acyclic on T

In this section, we focus on the special case of the relaxed (S, T) connectivity problem where the digraph G is acyclic on T. First, we show that this problem is at least as hard for approximation as the set covering problem. Then, we refine the algorithm of Section 6.1 to improve the approximation guarantee to $O(\log |S|)$ when the digraph is acyclic on T. We use the abbreviation SCP for the set covering problem.

For the hardness result, consider an instance I_{SC} of SCP with ground-set $U = \{u_1, \ldots, u_p\}$ and subsets $S_1, \ldots, S_q \subseteq U$. We can represent I_{SC} by a bipartite graph B whose vertex partition consists of U and $W = \{S_1, \ldots, S_q\}$; B has an edge between $u_i \in U$ and $S_j \in W$ iff $u_i \in S_j$ in the instance I_{SC}. To obtain an instance of relaxed (S,T) connectivity, we orient the edges of B from U to W, and then we add one new vertex t and the edges (S_j, t) with cost $c(S_j)$ for each subset S_j of the instance I_{SC}; we give a cost of zero to all other edges, and we fix $T = \{t\}$, $S = U$; note that each edge of positive cost has its head in T. This completes the construction. It can be seen that a feasible solution $\widehat{E}$ for the instance of relaxed (S,T) connectivity corresponds to a feasible solution of the instance I_{SC} of the same cost, by choosing a subset S_j of I_{SC} iff an edge (S_j, t) is in $\widehat{E}$.

Consider the refined approximation algorithm for relaxed (S,T) connectivity. For any vertex $\widehat{t} \in T$, when we compute an in-directed Steiner tree with root $\widehat{t}$ and terminal set S, then we only consider solution subgraphs such that all augmenting edges have heads at $\widehat{t}$; in other words, we ignore solution subgraphs that contain $s, \widehat{t}$ dipaths that use ≥ 2 augmenting edges for some $s \in S$. There may exist vertices $\widehat{t} \in T$ such that there exist no in-directed Steiner trees rooted at $\widehat{t}$ satisfying the above conditions; then, we give an infinite cost to the corresponding auxiliary edges $(\widehat{s}, \widehat{t})$ in the auxiliary digraph constructed by the algorithm. To see the correctness of this construction, consider an optimal solution $G^* = G_0 + E^*$ and any top vertex $\widehat{t}$ in the "decomposition" given by Lemma 42. The in-directed Steiner tree $J^{in}_{\widehat{t}}$ (of the decomposition) with root $\widehat{t}$ and terminal set S contains no other vertices of T, by property (i) of Lemma 42, hence, $J^{in}_{\widehat{t}}$ contains no augmenting edge with head in $T - \{\widehat{t}\}$.

We can compute an approximately min-cost directed Steiner tree of this special form by solving the following instance $\widehat{I}_{SC}$ of SCP. We take the ground-set in the instance $\widehat{I}_{SC}$ to be the set S (in the instance of relaxed (S,T) connectivity); moreover, for each edge e_j with head at $\widehat{t}$ (in the instance of relaxed (S,T) connectivity), we have a subset $S_j \subseteq S$ in the instance $\widehat{I}_{SC}$, where $S_j = \{s \in S : G_0 + e_j \text{ has an } s, \widehat{t} \text{ dipath}\}$.

A greedy algorithm for SCP gives an approximation guarantee of $O(\log |S|)$ for each of these instances of SCP (one for each vertex in T). Hence, by Proposition 43, the algorithm for relaxed (S,T) connectivity achieves an approximation guarantee of $O(\log |S|)$. This proves Theorem 5 in Section 1.3.

Acknowledgments. We thank our colleagues for useful comments. In particular, we thank Chaitanya Swamy and Jochen Könemann who were readers for the second author's thesis which contains many of the results of this paper.

References

[1] S.Arora and C.Lund, Hardness of Approximations. In *Approximation algorithms for NP-hard problems*, Ed. D.S.Hochbaum, PWS publishing co., Boston, 1996.

[2] S.Arora and M.Sudan, Improved Low-Degree Testing and its Applications. In *STOC*, pages 485–495, 1997.

[3] M.Charikar, C.Chekuri, T.-Y.Cheung, Z.Dai, A.Goel, S.Guha, and M.Li, Approximation algorithms for directed Steiner problems. In *SODA*, pages 192–200, 1998.

[4] C.Chekuri, G.Even, A.Gupta, and D.Segev, Set connectivity problems in undirected graphs and the directed Steiner network problem. *ACM Transactions on Algorithms*, 7(2), 2011. Preliminary version in Proc. 19th ACM-SIAM SODA, 532–541, 2008.

[5] J.Cheriyan, Combinatorial optimization on k-connected graphs. In Proc. *ICRTGC – ICM satellite conference*, Aug. 2010.

[6] J.Chuzhoy and S.Khanna, An $O(k^3 \log n)$-approximation algorithm for vertex connectivity survivable network design. In *FOCS*, pages 437–441. IEEE, 2009.

[7] G.Even, G.Kortsarz, and W.Slany, On network design problems: fixed cost flows and the covering Steiner problem. *ACM Transactions on Algorithms*, 1(1):74–101, 2005.

[8] J.Edmonds, Optimum branchings. *Journal of Research of National Bureau of Standards*, 71B(4):233–240, 1967.

[9] J.Fakcharoenphol and B.Laekhanukit, An $O(\log^2 k)$-approximation algorithm for the k-vertex connected spanning subgraph problem. In R.E. Ladner and C. Dwork, editors, *STOC*, pages 153–158. ACM, 2008.

[10] U.Feige, A threshold of $\ln n$ for approximating set cover. *J. ACM*, 45(4):634–652, 1998.

[11] L.Fleischer, K.Jain, and D.P.Williamson, An iterative rounding 2-approximation algorithm for the element connectivity problem. *Journal of Computer and System Sciences*, 72 (5), pp. 838–867, 2006. Preliminary version in *Proc. 42nd IEEE FOCS*, pp.339-347, 2001.

[12] A.Frank, Connectivity augmentation problems in network design, in *Mathematical Programming: State of the Art 1994*, (Eds.J.R.Birge and K.G.Murty), pp.34–63, The University of Michigan, Ann Arbor, MI, 1994.

[13] A.Frank, Increasing the rooted-connectivity of a digraph by one. *Mathematical Programming*, 84(3):565–576, 1999. Connectivity augmentation of networks: structures and algorithms (Budapest, 1994).

[14] A.Frank, Rooted k-connections in digraphs. *Discrete Applied Mathematics*, 157(6):1242–1254, 2009.

[15] A.Frank and T. Jordán, Minimal edge-coverings of pairs of sets. *J. Comb. Theory, Ser. B*, 65(1):73–110, 1995.

[16] A.Frank and E.Tardos, An application of submodular flows, *Linear Algebra and its Applications*, 114/115:329–348, 1989.329-348.

[17] G.Frederickson and J.JáJá, Approximation algorithms for several graph augmentation problems, *SIAM J. Computing*, 10(2):270–283, 1981.

[18] M.X. Goemans, A.V. Goldberg, S.A. Plotkin, D.B. Shmoys, E. Tardos, and D.P. Williamson, Improved approximation algorithms for network design problems. In *SODA*, pages 223–232, 1994.

[19] E. Halperin and R. Krauthgamer, Polylogarithmic inapproximability. In *STOC*, pages 585–594, 2003.

[20] C.S. Helvig, G. Robins, and A. Zelikovsky, An improved approximation scheme for the group Steiner problem. *Networks*, 37(1):8–20, 2001.

[21] S.Khuller, Approximation algorithms for finding highly connected subgraphs. In *Approximation algorithms for NP-hard problems*, Ed. D.S.Hochbaum, PWS publishing co., Boston, 1996.

[22] S.Khuller and B.Raghavachari, Improved approximation algorithms for uniform connectivity problems. *Journal of Algorithms* 21, 434–450, 1996.

[23] G.Kortsarz and Z.Nutov, Approximating node-connectivity problems via set covers. *Algorithmica*, 37(2):75–92, 2003.

[24] G.Kortsarz and Z.Nutov, Approximating k-node connected subgraphs via critical graphs. *SIAM J. Comput.*, 35(1):247–257, 2005.

[25] B.Laekhanukit, Approximation algorithms for (S,T)-connectivity problems. M.Math. Thesis, University of Waterloo, 2010.

[26] Z.Nutov, An almost $O(\log k)$-approximation for k-connected subgraphs. In Claire Mathieu, editor, *SODA*, pages 912–921. SIAM, 2009.

[27] Z.Nutov, Approximating minimum cost connectivity problems via uncrossable bifamilies and spider-cover decompositions. In *FOCS*, pages 417–426. IEEE, 2009.

[28] Z.Nutov, Approximating minimum-cost edge-covers of crossing biset-families. http://www.openu.ac.il/home/nutov/publications.html Manuscript, submitted for journal publication, May 2010. Preliminary version in [26].

[29] R. Ravi and D.P. Williamson, An approximation algorithm for minimum-cost vertex-connectivity problems. *Algorithmica*, 18(1):21–43, 1997. Preliminary version in Proc. 6th ACM-SIAM SODA, 332–341, 1995. Erratum. *Algorithmica*, 34:98–107, 2002.

[30] R.Raz and M.Safra, A Sub-Constant Error-Probability Low-Degree Test, and a Sub-Constant Error-Probability PCP Characterization of NP. In *STOC*, pages 475–484, 1997.

[31] A. Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency*. Algorithms and Combinatorics, Vol. 24. Springer, Berlin, 2003.

[32] V.V.Vazirani, *Approximation Algorithms*. Springer, Berlin, 2001.

[33] L.A. Végh and A.A. Benczúr, Primal-dual approach for directed vertex connectivity augmentation and generalizations. *ACM Transactions on Algorithms*, 4(2), 2008.

[34] D.P.Williamson and D.B.Shmoys, *The Design of Approximation Algorithms*. Cambridge University Press, 2010.